LECTURE 40

(40.1)

I. Spacetime metric for a homogeneous isotropic medium

II. Cartan-Misner calculus of the curvature induced by a homogeneous isotropic medium

III Ricci tensor, curvature invariant, and the Einstein tensor

In MTW read Box 14.5, § 27.8 (P 728-730), § 32.4 (P 851-852)
In the PDF scan of

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The Flatter Regions of Newman, Unti, and Tamburino's Generalized Schwarzschild Space*

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(which given as an augmentation to LECTURE 38) read the Appendix, page "935-936".

Nota bene: That article by Misner, in particular its Appendix, revolutionized mathematical gravitation physics.

(40.2)

I. Spacetime metric for a homogeneous isotropic medium

Astrophysical observation attest to the fact that the universe on a large scale (i.e. $>10^8$ light years) is filled with a homogeneous and isotropic fluid consisting of clusters of galaxies, dark matter, dark energy, and radiative energies of various types.

The density and pressure of this fluid is time dependent. Einstein's geometric law of gravitation establishes such a cosmic fluid as the cause of a time-dependent geometry mathematized by the metric tensor field,

$$g=(ds)^2=-dt^2+a^2(t)\underbrace{\left[d\chi^2+\sin^2\chi\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)\right]}_{\overset{\shortparallel}{d\sigma^2}}. \qquad (40.1)$$

Remark (From 2-sphere to 3-sphere)

Here $d\sigma^2$ is the metric tensor field on the (sub)manifold of all those events that occur at fixed cosmic time $t=$ const. The interval between any neighboring events is purely spacelike, and $d\sigma^2$ is the Pythagorian theorem applied to these events in an infinitesimal neighborhood surrounding any coordinatized point P mapped with coordinates $\chi(P)$, $\theta(P)$, $\varphi(P)$.

The metric tensor field $d\sigma^2$ is that of a manifold having the the geometry of a three-sphere. There are several mathematical method leading to this conclusion, and the method of a submanifold imbedded in a higher dimensional Euclidean space is one of them. (40.3)

Being mapped out by its rectilinear orthogonal Cartesian coordinates (x^1, x^2, x^3, x^4), a Euclidean space of 4-d accommodates a three-sphere as easily as one of 3-d accommodates a two-sphere.

A three-sphere is the locus of points (x^1, x^2, x^3, x^4) that satisfy the rotationally invariant constraint

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = a^2$$

The metric tensor on this 3-d manifold is

$(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2$ with the restriction that

$$0 = d\left[(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2\right] = 2\left(x^1\,dx^1 + x^2 dx^2 + x^3 dx^3 + x^4 dx^4\right). \qquad (40.2)$$

The metric with this constraint is exhibited with greater unit economy by means of spherical coordinates in 4-d Euclidean space:

$$x^1 = a \sin\chi \;\sin\theta \cos\varphi$$
$$x^2 = a \sin\chi \;\sin\theta \sin\varphi$$
$$x^3 = a \sin\chi \;\cos\theta$$
$$x^4 = a \cos\chi$$

The metric in terms of the coordinates is

$$(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 = (da)^2 + a^2\left[(d\chi)^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]$$

The imposition of the constraint, Eq. (40.2), on the 3-d surface of the 3-sphere implies that

$$\boxed{\left.(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2\right|_{a=\text{fixed}} = a^2\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right] \equiv a^2 d\sigma^2}$$

is the metric on the three-sphere of radius a, whatever it is as a function of physical time.

I. Spacetime metric for a homogeneous isotropic medium (continued) (40.4)

The requirement that the spacetime metric Eq. (40.1) satisfy the Einstein field equations leads to an ordinary differential equation [Eq. (27.39) in MTW] for $a(t)$. It is obtained from the Einstein tensor calculated on pages 40.13 of these notes. That equation mathematizes the dynamics of the geometry of the universe.

When the pressure in the fluid vanishes then $a(t)$ – by means of Eq. (32.12) in MTW – mathematizes the dynamics of a homogeneous spherical star in free fall collapse.

(40.5)

II. Cartan-Misner calculation of curvature induced by the metric

$$(ds)^2 = -dt^2 + a^2(t)\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]$$
$$= -(\hat\omega^t)^2 + (\hat\omega^\chi)^2 + (\hat\omega^\theta)^2 + (\hat\omega^\varphi)^2$$

Warning: From now on, until further note, drop — for typographical efficiency — the hat on the math symbols that refer to physical (i.e. orthonormal) bases. Thus $\hat\omega^\mu \longrightarrow \omega^\mu$, $\hat\Omega^\mu{}_\nu \longrightarrow \Omega^\mu{}_\nu$, etc.

Step 1

Establish the 1-forms of the basis dual to the o.n. basis

$$\omega^t = dt$$
$$\omega^\chi = a(t)\, d\chi$$
$$\omega^\theta = a(t)\sin\chi$$
$$\omega^\varphi = a(t)\sin\chi\,\sin\theta\, d\varphi$$

Step 2

Determine the connection 1-forms $\omega^\mu{}_\nu$ by solving the system of linear equations obtained from the 1st structure equation and from metric compatibility with $dg_{\mu\nu}=0$:

$$d\omega^\mu = \omega^\gamma \wedge \omega^\mu{}_\gamma$$

$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$

(a) Calculate the exterior derivatives and reexpress the result in terms of the o.n. basis elements: (40.6)

(i) $d\omega^t = 0$

(ii) $d\omega^\chi = \dot{a}\, dt \wedge d\chi$
$= \frac{\dot{a}}{a}\, \omega^t \wedge \omega^\chi$

(iii) $d\omega^\theta = \dot{a}\, dt \wedge \sin\chi\, d\theta + a\cos\chi\, d\chi \wedge d\theta$
$= \frac{\dot{a}}{a}\, \omega^t \wedge \omega^\theta + \frac{1}{a}\frac{\cos\chi}{\sin\chi}\, \omega^\chi \wedge \omega^\theta$

(iv) $d\omega^\varphi = \dot{a}\, dt \wedge \sin\chi \sin\theta\, d\varphi + a\cos\chi\sin\theta\, d\chi \wedge d\varphi + a\sin\chi\cos\theta\, d\theta \wedge d\varphi$
$= \frac{\dot{a}}{a}\, \omega^t \wedge \omega^\varphi + \frac{1}{a}\frac{\cos\chi}{\sin\chi}\, \omega^\chi \wedge \omega^\varphi + \frac{1}{a\sin\chi}\frac{\cos\theta}{\sin\theta}\, \omega^\theta \wedge \omega^\varphi$

(b) The connection 1-forms $\omega^\mu{}_\gamma$ are determined uniquely by

$$\omega^\gamma \wedge \omega^\mu{}_\gamma = d\omega^\mu; \quad \mu = 0, 1, 2, 3 \qquad (40.1)$$

together with the metric compatibility condition

$$\omega_{\mu\gamma} = -\omega_{\gamma\mu}. \qquad (40.2)$$

The solution to the inhomogeneous Eq. (40.1) is not unique. A solution $\{\omega^\mu{}_\gamma\}$ can always be augmented by an arbitrary solution to the homogeneous equation

$$\omega^\gamma \wedge \omega^\mu{}_\gamma = 0; \quad \mu = 0, 1, 2, 3$$

The arbitrariness is resolved by the compatibility requirement Eq. (40.2). Eqs. (40.1) and (40.2) yield a unique solution for the to-be-determined connection 1-forms $\{\omega^\mu{}_\gamma\}$.

There exists a very economical algorithm to achieve this goal. The structure diagram of this algorithm is depicted on page 40. after it has been illustrated by the ensuing calculations.

(i) For $\mu = t$ the equation to be solved, $\omega^\gamma \wedge \omega^t{}_\gamma = d\omega^t$, is

$$\omega^t \wedge \omega^t{}_t + \omega^x \wedge \omega^t{}_x + \omega^\theta \wedge \omega^t{}_\theta + \omega^\varphi \wedge \omega^t{}_\varphi = 0 \qquad (40.7)$$

The solution set $\{\omega^t{}_\gamma\}_{\gamma=0}^{3}$ is as follows:

(1) $\omega^t{}_t = -\omega_{tt} = 0 \implies \omega^t{}_t = 0$

Assume simplest particular solution set:

(2) $\omega^t{}_x = (?^1)\, \omega^x$ (i.2)

(3) $\omega^t{}_\theta = (??^1)\, \omega^\theta$ (i.3)

(4) $\omega^t{}_\varphi = (???^1)\, \omega^\varphi$ (i.4)

Whether this assumption is valid or not is born out by requiring its consistency when $\omega^t{}_\gamma$ is compared with the other $\omega^\mu{}_\gamma$'s.

Remark

Instead of that single term assumption, some metrics require a two-term assumption in order to arrive at the connection 1-forms $\omega^\mu{}_\gamma$, and hence the curvature. The Kerr geometry of a rotating black hole is one of these.

(ii) For $\mu = x$ the equation to be solved, $\omega^\gamma \wedge \omega^x{}_\gamma = d\omega^x$, is $\omega^t \wedge \omega^x{}_t + \omega^x \wedge \omega^x{}_x + \omega^\theta \wedge \omega^x{}_\theta + \omega^\varphi \wedge \omega^x{}_\varphi = \frac{\dot a}{a}\, \omega^t \wedge \omega^x$

Again, based on the one-term assumption

(1) $\omega^x{}_t = \frac{\dot a}{a}\omega^x + (?^2)\, \omega^t$ (ii.1)

Also (2) $\omega^x{}_x = \omega_{xx} = 0 \implies \omega^x{}_x = 0$

and again

(3) $\omega^{\chi}{}_{\theta} = (???^2)\,\omega^{\theta}$ (ii.3) 40.8

(4) $\omega^{\chi}{}_{\varphi} = (????^2)\,\omega^{\varphi}$ (ii.4)

Noting that

$$\omega^{\chi}{}_{t} = \omega_{\chi t} = -\omega_{t\chi} = \omega^{t}{}_{\chi}$$

equate Eq. (ii.1) to (i.2)

$$\frac{\dot a}{a}\,\omega^{\chi} + (?^2)\,\omega^{t} = (?^1)\,\omega^{\chi} + 0\cdot\omega^{t}$$

Consequently, $(?^2) = 0$ and $\frac{\dot a}{a} = (?^1)$.

Thus

(2) $\boxed{\omega^{\chi}{}_{t} = \frac{\dot a}{a}\,\omega^{\chi} = \omega^{t}{}_{\chi}}$

(iii) For $\mu = \theta$ the equation to be solved, $\omega^{\gamma}\wedge\omega^{\theta}{}_{\gamma} = d\omega^{\theta}$, is

$$\omega^{t}\wedge\omega^{\theta}{}_{t} + \omega^{\chi}\wedge\omega^{\theta}{}_{\chi} + \omega^{\theta}\wedge\omega^{\theta}{}_{\theta} + \omega^{\varphi}\wedge\omega^{\theta}{}_{\varphi} =$$

$$= \frac{\dot a}{a}\,\omega^{t}\wedge\omega^{\theta} + \frac{1}{a}\frac{\cos\chi}{\sin\chi}\,\omega^{\chi}\wedge\omega^{\theta} + 0\cdot\omega^{\varphi}\wedge\omega^{\theta}$$

Again, based on the one-term assumption

(1) $\omega^{\theta}{}_{t} = \frac{\dot a}{a}\,\omega^{\theta} + (?^2)\,\omega^{t}$ (iii.1)

(2) $\omega^{\theta}{}_{\chi} = \frac{1}{a}\frac{\cos\chi}{\sin\chi}\,\omega^{\theta} + (??^2)\,\omega^{\chi}$ (iii.2)

Also (3) $\omega^{\theta}{}_{\theta} = \omega_{\theta\theta} = 0 \Longrightarrow \omega^{\theta}{}_{\theta} = 0$

and again

(4) $\omega^{\theta}{}_{\varphi} = (???^2)\,\omega^{\varphi}$ (iii.3)

Noting that

$$\omega^{\theta}{}_{t} = \omega_{\theta t} = -\omega_{t\theta} = \omega^{t}{}_{\theta}$$

$$\omega^{\theta}{}_{\chi} = \omega_{\theta\chi} = -\omega_{\chi\theta} = -\omega^{\chi}{}_{\theta},$$ 40.9

equate Eq. (iii.1) to Eq. (i.3):

$$\left(\omega^{\theta}{}_{t}\right) = \frac{\dot a}{a}\omega^{\theta} + \underbrace{(?^{2})}_{\text{zero}}\omega^{t} = (??^{1})\,\omega^{\theta}\left(=\omega^{t}{}_{\theta}\right)$$

and find

$$\boxed{\omega^{\theta}{}_{t} = \omega^{t}{}_{\theta} = \frac{\dot a}{a}\omega^{\theta}}$$

Similarly equate Eq. (iii.2) to (−)Eq. (ii.3)

$$\left(\omega^{\theta}{}_{\chi} =\right) \frac{1}{a}\frac{\cos\chi}{\sin\chi}\,\omega^{\theta} + \underbrace{(??^{2})}_{\text{zero}}\omega^{\chi} = -(??^{2})\,\omega^{\theta}\left(= -\omega^{\chi}{}_{\theta}\right)$$

and find

$$\boxed{\omega^{\theta}{}_{\chi} = -\omega^{\chi}{}_{\theta} = \frac{\cot\theta}{a}\,\omega^{\theta}}$$

(iv) For $\mu = \varphi$ the equation to be solved, $\omega^{\gamma}\wedge\omega^{\varphi}{}_{\gamma} = d\omega^{\varphi}$, is $\omega^{t}\wedge\omega^{\varphi}{}_{t} + \omega^{\chi}\wedge\omega^{\varphi}{}_{\chi} + \omega^{\theta}\wedge\omega^{\varphi}{}_{\theta} + \omega^{\varphi}\wedge\omega^{\varphi}{}_{\varphi} =$

$$= \frac{\dot a}{a}\,\omega^{t}\wedge\omega^{\varphi} + \frac{\cot\chi}{a}\,\omega^{\chi}\wedge\omega^{\varphi} + \frac{\cot\theta}{a\sin\chi}\,\omega^{\theta}\wedge\omega^{\varphi}$$

Again, based on the one-term assumption

(1) $\omega^{\varphi}{}_{t} = \frac{\dot a}{a}\omega^{\varphi} + (?^{3})\,\omega^{t}$ (iv.1)

(2) $\omega^{\varphi}{}_{\chi} = \frac{\cot\chi}{a}\omega^{\varphi} + (??^{3})\,\omega^{\chi}$ (iv.2)

(3) $\omega^{\varphi}{}_{\theta} = \frac{\cot\theta}{a\sin\chi}\,\omega^{\varphi} + (???^{3})\,\omega^{\theta}$ (iv.3)

Also

(4) $\omega^{\varphi}{}_{\varphi} = \omega_{\varphi\varphi} = 0 \implies \omega^{\varphi}{}_{\varphi} = 0$

Noting that (40.10)

$$\omega^{\varphi}{}_{t} = \omega_{\varphi t} = -\omega_{t\varphi} = \omega^{t}{}_{\varphi}$$
$$\omega^{\varphi}{}_{\chi} = \omega_{\varphi\chi} = -\omega_{\chi\varphi} = -\omega^{\chi}{}_{\varphi}$$
$$\omega^{\varphi}{}_{\theta} = \omega_{\varphi\theta} = -\omega_{\theta\varphi} = -\omega^{\theta}{}_{\varphi}$$

Equate Eq. (iv.1) to Eq. (i.4)

$$\left(\omega^{\varphi}{}_{t} =\right) \frac{\dot{a}}{a}\omega^{\varphi} + \underbrace{\left(?^3\right)}_{\text{zero}}\omega^{t} = \left(???^4\right)\omega^{\varphi}\left(= \omega^{t}{}_{\varphi}\right)$$

and find

$$\boxed{\omega^{\varphi}{}_{t} = \omega^{t}{}_{\varphi} = \frac{\dot{a}}{a}\omega^{\varphi}}$$

Equate Eq. (iv.2) to (−) Eq. (ii.4)

$$\left(\omega^{\varphi}{}_{\chi} =\right) \frac{\cot\chi}{a}\omega^{\varphi} + \underbrace{\left(??^3\right)}_{\text{zero}}\omega^{\chi} = -\left(???^2\right)\omega^{\varphi}\left(= -\omega^{\chi}{}_{\varphi}\right)$$

and find

$$\boxed{\omega^{\varphi}{}_{\chi} = -\omega^{\chi}{}_{\varphi} = \frac{\cot\chi}{a}\omega^{\varphi}}$$

Equate Eq. (iv.3) to (−) Eq. (iii.4)

$$\left(\omega^{\varphi}{}_{\theta} =\right) \frac{\cot\theta}{a\sin\chi}\omega^{\varphi} + \underbrace{\left(???^3\right)}_{\text{zero}}\omega^{\theta} = -\left(???^2\right)\omega^{\varphi}\left(= -\omega^{\theta}{}_{\varphi}\right)$$

and find

$$\boxed{\omega^{\varphi}{}_{\theta} = -\omega^{\theta}{}_{\varphi} = \frac{\cot\theta}{a\sin\chi}\omega^{\varphi}}$$

SUMMARY

The computed connection 1-forms are

$$\omega^t{}_t = 0 \qquad \omega^t{}_\chi = \frac{\dot a}{a}\,\omega^\chi = \omega^\chi{}_t \qquad \omega^t{}_\theta = \frac{\dot a}{a}\,\omega^\theta = \omega^\theta{}_t \qquad \omega^t{}_\varphi = \frac{\dot a}{a}\,\omega^\varphi = \omega^\varphi{}_t \tag{40.11}$$

$$\omega^\chi{}_\chi = 0 \qquad \omega^\chi{}_\theta = -\frac{\cot\chi}{a}\,\omega^\theta = -\omega^\theta{}_\chi \qquad \omega^\chi{}_\varphi = -\frac{\cot\chi}{a}\,\omega^\varphi = -\omega^\varphi{}_\chi$$

$$\omega^\theta{}_\theta = 0 \qquad \omega^\theta{}_\varphi = -\frac{\cot\theta}{a\sin\chi}\,\omega^\varphi = -\omega^\varphi{}_\theta$$

$$\omega^\varphi{}_\varphi = 0$$

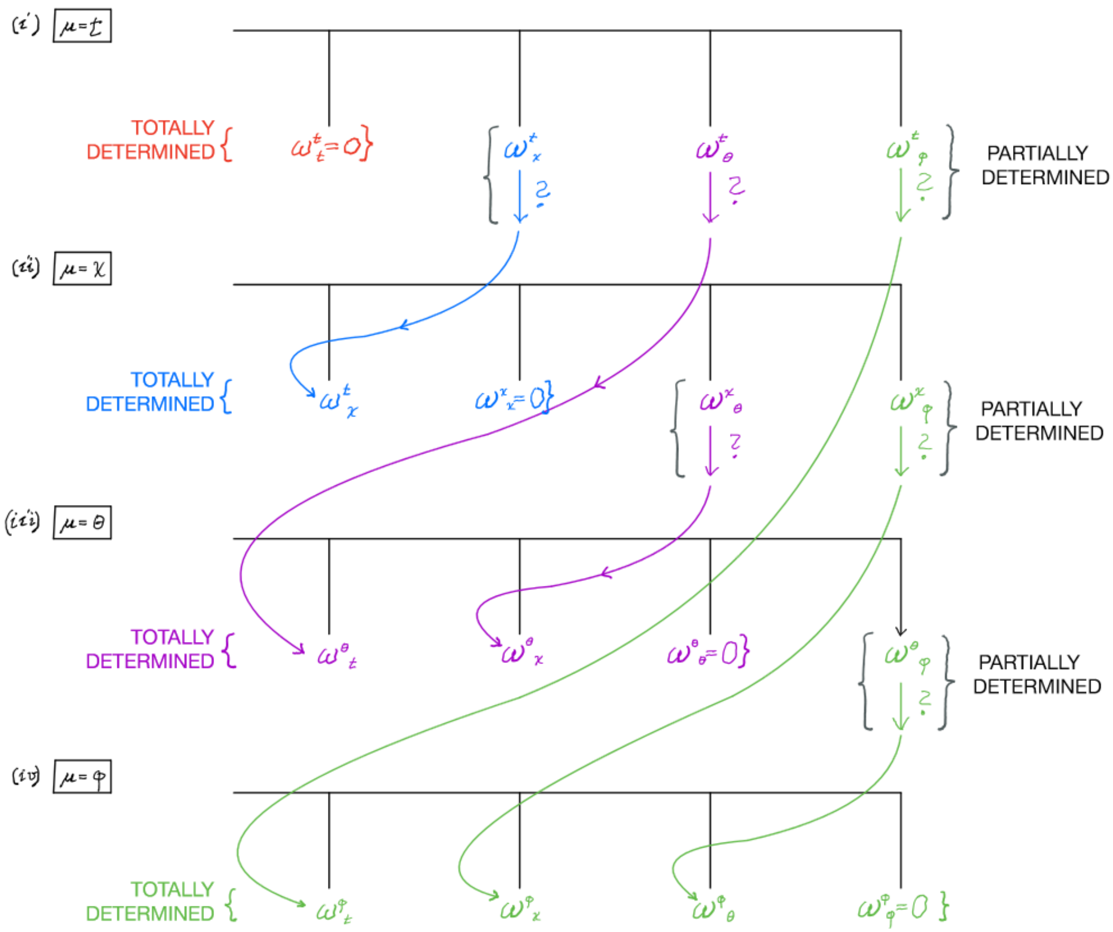

Figure 40.1

40.12

Structure diagram of the algorithm for computing the connection one-forms $\{\omega^\mu{}_\nu\}$ from

$$\omega^\gamma \wedge \omega^\mu{}_\gamma = d\omega^\mu$$
and
$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$

Step 3

Calculate the curvature 2-forms

$$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\gamma \wedge \omega^\gamma{}_\nu$$

$$\Omega^t{}_\chi = d\omega^t{}_\chi + \underbrace{\omega^t{}_t \wedge \omega^t{}_\chi}_{\text{zero}} + \underbrace{\omega^t{}_\chi \wedge \omega^\chi{}_\chi}_{\text{zero}} + \omega^t{}_\theta \wedge \omega^\theta{}_\chi + \omega^t{}_\varphi \wedge \omega^\varphi{}_\chi$$

$$= d\left(\frac{\dot a}{a}\omega^\chi\right) + 0 + 0 + (\;)\underbrace{\omega^\theta \wedge \omega^\theta}_{\text{zero}} + (\;)\underbrace{\omega^\varphi \wedge \omega^\varphi}_{\text{zero}}$$

$$= d\left(\frac{\dot a}{a} a\, d\chi\right)$$

$$= \ddot a\, dt \wedge d\chi$$

$$\boxed{\Omega^t{}_\chi = \frac{\ddot a}{a}\omega^t \wedge \omega^\chi} \qquad \boxed{R^t{}_{\chi t \chi} = \frac{\ddot a}{a}}$$

$$\Omega^t{}_\theta = d(\omega^t{}_\theta) + \underbrace{\omega^t{}_t \wedge \omega^t{}_\theta}_{\text{zero}} + \omega^t{}_\chi\, \omega^\chi{}_\theta + \underbrace{\omega^t{}_\theta\, \omega^\theta{}_\theta}_{\text{zero}} + \omega^t{}_\varphi \wedge \omega^\varphi{}_\theta$$

$$= d\left(\frac{\dot a}{a}\omega^\theta\right) + 0 + \frac{\dot a}{a}\omega^\chi \wedge (-)\frac{\cot\chi}{a}\omega^\theta + (\;)\omega^\varphi \wedge \omega^\varphi$$

$$= d\left(\frac{\dot a}{a} a \sin\chi\, d\theta\right) + \frac{\dot a}{a} a\, d\chi \wedge (-)\frac{\cos\chi}{a\sin\chi} a \sin\chi\, d\theta$$

$$= \ddot a \sin\chi\, dt \wedge d\theta + \underbrace{\dot a \cos\chi\, d\chi \wedge d\theta - \dot a \cos\chi\, d\chi \wedge d\theta}_{\text{cancel}}$$

40.13

$$\boxed{\Omega^t{}_\theta = \frac{\ddot{a}}{a}\,\omega^t \wedge \omega^\theta} \qquad \boxed{R^t{}_{\theta t \theta} = \frac{\ddot{a}}{a}}$$

Similarly

$$\boxed{\Omega^t{}_\varphi = \frac{\ddot{a}}{a}\,\omega^t \wedge \omega^\varphi} \qquad \boxed{R^t{}_{\varphi t \varphi} = \frac{\ddot{a}}{a}}$$

$$\Omega^\chi{}_\theta = d(\omega^\chi{}_\theta) + \omega^\chi{}_t \wedge \omega^t{}_\theta + \underbrace{\omega^\chi{}_\chi \wedge \omega^\chi{}_\theta}_{\text{zero}} + \underbrace{\omega^\chi{}_\theta \wedge \omega^\theta{}_\theta}_{\text{zero}} + \omega^\chi{}_\varphi \wedge \omega^\varphi{}_\theta$$

$$= d\left(-\frac{\cos\chi}{a\sin\chi}\,a\sin\chi\,d\theta\right) + \frac{\dot{a}}{a}\,a\,d\chi \wedge \frac{\dot{a}}{a}\,a\sin\chi\,d\theta + 0 + 0 + (\ )\,\omega^\varphi \wedge \omega^\varphi$$

$$= \sin\chi\,d\chi \wedge d\theta + \left(\frac{\dot{a}}{a}\right)^2 \omega^\chi \wedge \omega^\theta$$

$$= \frac{1}{a^2}\,\omega^\chi \wedge \omega^\theta + \left(\frac{\dot{a}}{a}\right)^2 \omega^\chi \wedge \omega^\theta$$

$$\boxed{\Omega^\chi{}_\theta = \frac{1+\dot{a}^2}{a^2}\,\omega^\chi \wedge \omega^\theta} \qquad \boxed{R^\chi{}_{\theta\chi\theta} = \frac{1+\dot{a}^2}{a^2}}$$

$$\Omega^\chi{}_\varphi = d(\omega^\chi{}_\varphi) + \omega^\chi{}_t \wedge \omega^t{}_\varphi + \underbrace{\omega^\chi{}_\chi \wedge \omega^\chi{}_\varphi}_{\text{zero}} + \omega^\chi{}_\theta \wedge \omega^\theta{}_\varphi + \underbrace{\omega^\chi{}_\varphi \wedge \omega^\varphi{}_\varphi}_{\text{zero}}$$

$$= d\left(-\frac{\cos\chi}{a\,\cancel{\sin\chi}}\,a\,\cancel{\sin\chi}\,\sin\theta\,d\varphi\right) + \frac{\dot{a}}{a}\,a\,d\chi \wedge \frac{\dot{a}}{a}\,a\sin\chi\sin\theta\,d\varphi$$

$$+ (-)\frac{\cos\chi}{\cancel{a\sin\chi}}\,\cancel{a\sin\chi}\,d\theta \wedge (-)\frac{\cos\theta}{\cancel{\sin\theta}}\frac{1}{\cancel{a\sin\chi}}\,\cancel{a\sin\chi\sin\theta}\,d\varphi$$

$$= \sin\chi\sin\theta\,d\chi \wedge d\varphi - \cos\chi\cos\theta\,d\theta \wedge d\varphi + \left(\frac{\dot{a}}{a}\right)^2 a\,d\chi \wedge a\sin\chi\sin\theta\,d\varphi$$

cancel

$$+ \cos\chi\cos\theta\,d\theta \wedge d\varphi$$

$$\boxed{\Omega^\chi{}_\varphi = \frac{1+\dot{a}^2}{a^2}\,\omega^\chi \wedge \omega^\varphi} \qquad \boxed{R^\chi{}_{\varphi\chi\varphi} = \frac{1+\dot{a}^2}{a^2}}$$

Similarly

$$\boxed{\Omega^\theta{}_\varphi = \frac{1+\dot{a}^2}{a^2}\,\omega^\theta \wedge \omega^\varphi} \qquad \boxed{R^\theta{}_{\varphi\theta\varphi} = \frac{1+\dot{a}^2}{a^2}}$$

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Summary

The non-zero independent curvature components relative to an o.n. frame are

$$R^{t}{}_{\chi t \chi} = R^{t}{}_{\theta t \theta} = R^{t}{}_{\varphi t \varphi} = \frac{\ddot{a}}{a}$$

$$R^{\chi}{}_{\theta \chi \theta} = R^{\chi}{}_{\varphi \chi \varphi} = R^{\theta}{}_{\varphi \theta \varphi} = \frac{1+\dot{a}^2}{a^2}$$

Ricci tensor components : $R^{\alpha}{}_{\mu \alpha \nu} = R_{\mu\nu}$ are

$$R_{tt} = R^{\chi}{}_{t \chi t} + R^{\theta}{}_{t \theta t} + R^{\varphi}{}_{t \varphi t}$$
$$= -\frac{\ddot{a}}{a} - \frac{\ddot{a}}{a} - \frac{\ddot{a}}{a} = -3\frac{\ddot{a}}{a}$$

$$R_{\chi\chi} = R^{t}{}_{\chi t \chi} + R^{\theta}{}_{\chi \theta \chi} + R^{\varphi}{}_{\chi \varphi \chi}$$
$$= \frac{\ddot{a}}{a} + \frac{1+\dot{a}^2}{a^2} + \frac{1+\dot{a}^2}{a^2} = \frac{\ddot{a}}{a} + 2\,\frac{1+\dot{a}^2}{a^2}$$

$$R_{\theta\theta} = R^{t}{}_{\theta t \theta} + R^{\chi}{}_{\theta \chi \theta} + R^{\varphi}{}_{\theta \varphi \theta}$$
$$= \frac{\ddot{a}}{a} + \frac{1+\dot{a}^2}{a^2} + \frac{1+\dot{a}^2}{a^2} = \frac{\ddot{a}}{a} + 2\,\frac{1+\dot{a}^2}{a^2}$$

$$R_{\varphi\varphi} = R^{t}{}_{\varphi t \varphi} + R^{\chi}{}_{\varphi \chi \varphi} + R^{\theta}{}_{\varphi \theta \varphi}$$
$$= \frac{\ddot{a}}{a} + \frac{1+\dot{a}^2}{a^2} + \frac{1+\dot{a}^2}{a^2} = \frac{\ddot{a}}{a} + 2\,\frac{1+\dot{a}^2}{a^2}$$

$$[R_{\mu\nu}] = \begin{bmatrix} R_{tt} & & & \\ & R_{\chi\chi} & 0 & \\ & 0 & R_{\theta\theta} & \\ & & & R_{\varphi\varphi} \end{bmatrix} = \begin{bmatrix} -3\frac{\ddot{a}}{a} & & & \\ & \frac{\ddot{a}}{a}+2\frac{1+\dot{a}^2}{a^2} & 0 & \\ & 0 & \frac{\ddot{a}}{a}+2\frac{1+\dot{a}^2}{a^2} & \\ & & & \frac{\ddot{a}}{a}+2\frac{1+\dot{a}^2}{a^2} \end{bmatrix}$$

Curvature invariant

40.15

$$R \equiv R^t{}_t + R^x{}_x + R^\theta{}_\theta + R^\varphi{}_\varphi$$

$$= 3\frac{\ddot{a}}{a} + 3\left(\frac{\ddot{a}}{a} + 2\,\frac{1+\dot{a}^2}{a^2}\right)$$

$$\boxed{R = 6\left(\frac{\ddot{a}}{a} + \frac{1+\dot{a}^2}{a^2}\right)}$$

Components of the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R$$

$$G_{tt} = R_{tt} - \tfrac{1}{2} g_{tt} R$$

$$= -3\frac{\ddot{a}}{a} + \tfrac{1}{2} \times 6\left(\frac{\ddot{a}}{a} + \frac{1+\dot{a}^2}{a^2}\right)$$

$$\boxed{G_{tt} = 3\,\frac{1+\dot{a}^2}{a^2}}$$

$$G_{xx} = R_{xx} - \tfrac{1}{2} g_{xx} R$$

$$= \frac{\ddot{a}}{a} + 2\,\frac{1+\dot{a}^2}{a^2} - \tfrac{1}{2}\,6\left(\frac{\ddot{a}}{a} + \frac{1+\dot{a}^2}{a^2}\right)$$

$$= -2\,\frac{\ddot{a}}{a} - \frac{1+\dot{a}^2}{a^2}$$

$$\boxed{G_{xx} = -\left(2\,\frac{\ddot{a}}{a} + \frac{1+\dot{a}^2}{a^2}\right)}$$

$$G_{\theta\theta} = R_{\theta\theta} - \tfrac{1}{2} g_{\theta\theta} R$$

$$\boxed{G_{\theta\theta} = -\left(2\,\frac{\ddot{a}}{a} + \frac{1+\dot{a}^2}{a^2}\right)}$$

$$\boxed{G_{\varphi\varphi} = -\left(2\,\frac{\ddot{a}}{a} + \frac{1+\dot{a}^2}{a^2}\right)}$$

All off diagonal elements are zero.