LECTURE 5

5.1

Minkowski (a.k.a. Lorentz) spacetime as a vector space

1.) Its observational basis

a. Events

b. Vector as an equivalence class of Lorentz xformation related measurement

c. The homogeneity and vector space structure of Minkowski spacetime

d. Einstein's summation convention

2.) Applications

a. Four-velocity; four-acceleration

b. Wave propagation 4-vector

$\Big\}$ consigned to LECTURE 6

Reading Assignment:

[In MTW: Box 1.3, 2.4; Sect 2.1–2.3 . In T-W Box 7.1]

I. EVENTS

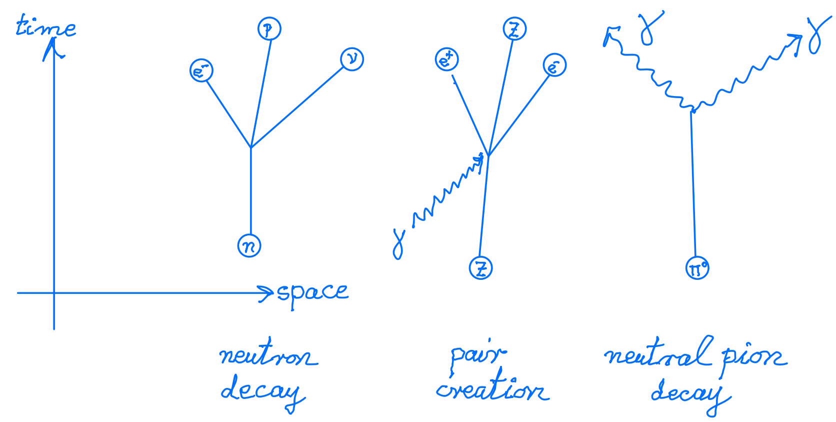
The basic building blocks of spacetime are <u>events</u>:

1.) Lightning struck our neighbor's chimney 30 feet above ground, 50 feet south, and 100 feet west of us at 12:30 pm, August 31, 2020.

2.) Two stars of a binary system merge into a single entity resulting in the Kepler super nova of 1604 at a distance of 20,000 light years in the direction with spherical spherical coordinates right ascension $17^h 30^m 42^s$ and declination $-21°29'$.

3.) The birth of Einstein on 3/14/1879 at latitude 48.4011°N and longitude 9.9876°E and 479 meters above sea level.

4.)



$\mathcal{X}_n$ $\qquad$ $\mathcal{X}_{create}$ $\qquad$ $\mathcal{X}_\pi$

Events are multifarious, be they in the realm of meteorology, astrophysics, labor and delivery, etc. However, each of them is a Lorentz invariant: Be it $\mathcal{X}_L, \mathcal{X}_S, \mathcal{X}_E, \dots, \mathcal{X}_\pi, \dots$ or $\mathcal{X}_{\dots}$, the existence of each of them is independent of any observer in any inertial frame. Moreover, human observers take quantitative cognizance of events in terms of four spacetime coordinates. These are measurements relative to the clocks and meter rods that comprise their respective inertial frames, say, S or $\bar{S}$ or $\bar{\bar{S}}$ or $\cdots$. For each event one has the following coordinate maps:

Event:		$\mathcal{X}$		
coordinate function ("map"):	x^μ	$\bar{x}^\mu$	$\bar{\bar{x}}^\mu$	$\mu = 0,1,2,3$
measured coordinate value:	$x^\mu(\mathcal{X})$	$\bar{x}^\mu(\mathcal{X})$	$\bar{\bar{x}}^\mu(\mathcal{X})$	
relative to frame:	S	$\bar{S}$	$\bar{\bar{S}}$	$\cdots$

Figure 5.1a: Spacetime coordinates of event $\mathcal{X}$ relative to frames $S, \bar{S}, \bar{\bar{S}} \cdots$.

Each event $\mathcal{X}$ has its four measured coordinate values

$$\{t(\mathcal{X}), x(\mathcal{X}), y(\mathcal{X}), z(\mathcal{X})\} \equiv \{x^0(\mathcal{X}), x^1(\mathcal{X}), x^2(\mathcal{X}), x^3(\mathcal{X})\} \equiv \{x^\mu(\mathcal{X})\}_{\mu=0}^{3}$$

relative to the observer in his frame S. Analogous mathematizations are done relative to the other frames $\bar{S}, \bar{\bar{S}}, \cdots$.

The above mathematization of the naturally occuring events $\{\mathcal{X}_L, \mathcal{X}_S, \mathcal{X}_E, \cdots, \mathcal{X}, \cdots\}$ is condensed into the following

DEFINITION

The mapping

$$\{\mathcal{X}\} \xrightarrow{\{x^\mu\}} R^4$$

$$\mathcal{X} \rightsquigarrow \{x^\mu(\mathcal{X}) : \mu = 0,1,2,3\}$$

is called the <u>coordinate system</u> (aka <u>coordinate chart</u>) of observer S.

One speaks of $\{x^\mu\}$ being a <u>coodinatization of the spacetime</u> events $\{\mathcal{X}\}$ relative to S, or simply spacetime being <u>coordinatized by</u> the coordinate system <u>$\{x^\mu : \mu = 0,1,2,3\}$</u> $\equiv \{x^\mu\}$.

Analogous statements hold for $\{\bar{x}^{\mu} : 0,1,2,3\}$ being the coordinate function or coordinate system relative to observer $\bar{S}$. Thus, in (i.e. relative to) different frames, an event $\mathcal{X}$ gets coordinatized in different ways:

$$\begin{array}{cccc} S & \bar{S} & \bar{\bar{S}} & \\ \{x^{\mu}(\mathcal{X})\}, & \{\bar{x}^{\mu}(\mathcal{X})\}, & \{\bar{\bar{x}}^{\mu}(\mathcal{X})\}, & \cdots \end{array}$$

In spite of this difference there is an equivalence relation between the different coordinatizations of a given event. This equivalence is made explicit by the Lorentz Lorentz transformations

$$\bar{\Lambda}: \bar{x}^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\bar{\mu}}{}_{\nu}\, x^{\nu}; \qquad \bar{\bar{\Lambda}}: \bar{\bar{x}}^{\mu} = \sum_{\bar{\nu}=0}^{3} \Lambda^{\bar{\bar{\mu}}}{}_{\bar{\nu}}\, \bar{x}^{\bar{\nu}}; \qquad \cdots$$

which is made explicit by Eq. (5.6) on page 5.6. It follows that the overlaping neighbor hood of the observers $S, \bar{S}, \bar{\bar{S}}, \ldots$, which surround the given event $\mathcal{X}$, form the following network of relationships

$$\begin{tikzcd} & \mathcal{X} \arrow[dl, "x^{\mu}"'] \arrow[d, "\bar{x}^{\mu}"] \arrow[dr, "\bar{\bar{x}}^{\mu}"] & & \\ x^{\mu}(\mathcal{X}) \arrow[r, "\bar{\Lambda}"] & \bar{x}^{\mu}(\mathcal{X}) \arrow[r, "\bar{\bar{\Lambda}}"] & \bar{\bar{x}}^{\mu}(\mathcal{X}) & \cdots \\ S & \bar{S} & \bar{\bar{S}} & \cdots \end{tikzcd}$$

Figure 5.1b: Parts of event $\mathcal{X}$ (e.g. the fission products resulting from the fissioning of an atomic nucleus) are present in the observer frames $S, \bar{S}, \bar{\bar{S}}, \ldots$. There they are recorded in terms of the Lorentz-transformation-related observed coordinate values $x^{\mu}(\mathcal{X})$, $\bar{x}^{\mu}(\mathcal{X})$, $\bar{\bar{x}}^{\mu}(\mathcal{X})$, ….

To every set of inertial frames, say $S, \bar{S}, \bar{\bar{S}}, \cdots$, there corresponds to each $\mathcal{X}$ the corresponding set of quadruple of numbers $\{x^\mu(\mathcal{X}), \bar{x}^\mu(\mathcal{X}), \bar{\bar{x}}^\mu(\mathcal{X}), \cdots\}$, the measured spacetime coordinates of the event $\mathcal{X}$ that occurred in $S, \bar{S}, \bar{\bar{S}}, \cdots$.

This correspondence is one-to-one and onto. The set $\{x^\mu(\mathcal{X}), \bar{x}^\mu(\mathcal{X}), \bar{\bar{x}}^\mu(\mathcal{X}), \cdots\}$ identifies in numerical form the event $\mathcal{X}$ regardless of the details of its localized internal dynamical spacetime structure. Thus, by writing

$$\mathcal{X} \doteq \{x^\mu(\mathcal{X}), \bar{x}^\mu(\mathcal{X}), \bar{\bar{x}}^\mu(\mathcal{X}), \cdots\}$$

one has put each event $\mathcal{X}$ into quantitative form. In this form one can take scalar multiples as well as sums and differences with other events.

II. VECTORS

The original conception of a vector arose (already with Euclid) from mathematizing the idea of a directed displacement (with Galileo and Newton). Consider the pair of directed spacetime displacements

$$\Delta\mathcal{X}_1 = \mathcal{X}_1 - \mathcal{X}_0 = \{x^\mu(\mathcal{X}_1) - x^\mu(\mathcal{X}_0), \bar{x}^\mu(\mathcal{X}_1) - \bar{x}^\mu(\mathcal{X}_0), \bar{\bar{x}}^\mu(\mathcal{X}_1) - \bar{\bar{x}}(\mathcal{X}_0), \cdots\} \equiv \{\Delta x_1^\mu, \Delta\bar{x}_1^\mu, \Delta\bar{\bar{x}}_1^\mu \cdots\}$$

and

$$\Delta\mathcal{X}_2 = \mathcal{X}_2 - \mathcal{X}_0 = \{x^\mu(\mathcal{X}_2) - x^\mu(\mathcal{X}_0), \bar{x}^\mu(\mathcal{X}_2) - \bar{x}^\mu(\mathcal{X}_0), \bar{\bar{x}}^\mu(\mathcal{X}_2) - \bar{\bar{x}}(\mathcal{X}_0), \cdots\} \equiv \{\Delta x_2^\mu, \Delta\bar{x}_2^\mu, \Delta\bar{\bar{x}}_2^\mu \cdots\}$$

where $\mathcal{X}_0$ is the reference event whose spacetime coordinates are zero $(x^\mu = \bar{x}^\mu = \bar{\bar{x}}^\mu \cdots = 0\,;\, \mu = 0,1,2,3)$ relative to all reference frames $S, \bar{S}, \bar{\bar{S}}, \cdots$.

Nota bene: Notice that <u>each</u> displacement is a displacement between <u>two</u> events, $\mathcal{X}_1$ and $\mathcal{X}_0$ for $\Delta\mathcal{X}_1$, and $\mathcal{X}_2$ and $\mathcal{X}_0$ for $\Delta\mathcal{X}_2$.

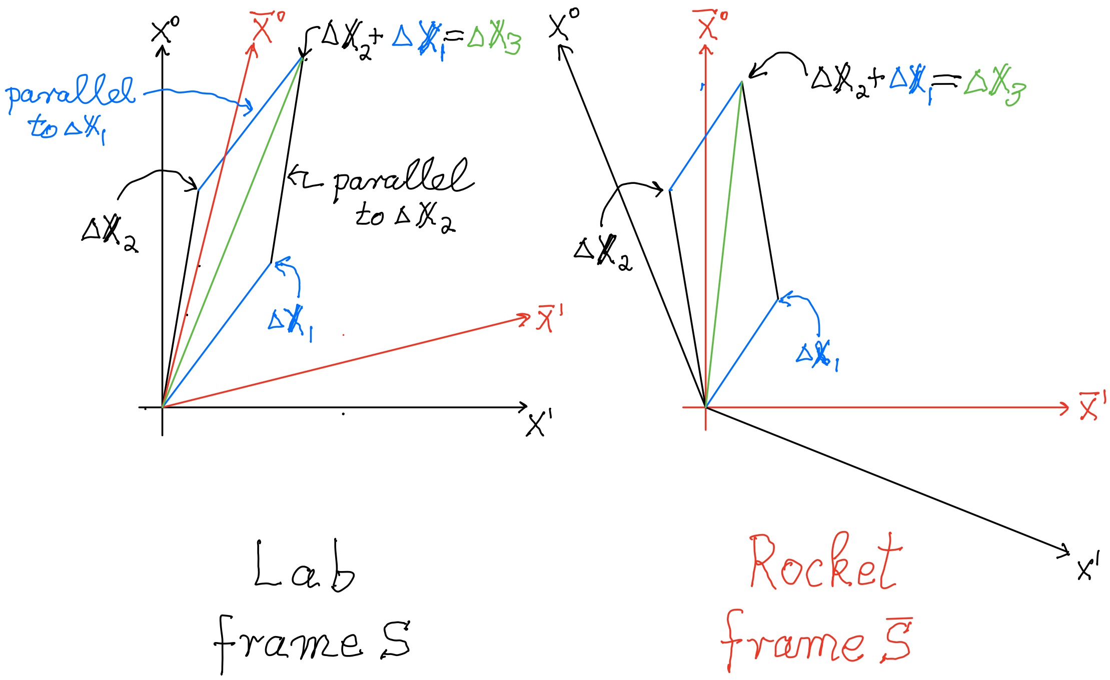


Figure 5.2 : Vector addition via parallelogram construction.

The Lorentz transformation of the displacement vector components between the LAB and the ROCKET frames is

$$\Delta \mathbb{X}_1 : \{\Delta \bar{x}_1^\mu\} = \Lambda(\{\Delta x_1^\mu\}) \qquad (5.1)$$

$$\Delta \mathbb{X}_2 : \{\Delta \bar{x}_2^\mu\} = \Lambda(\{\Delta x_2^\mu\}) \qquad (5.2)$$

$$\Delta \mathbb{X}_1 + \Delta \mathbb{X}_2 = \Delta \mathbb{X}_3 : \{\Delta \bar{x}_3^\mu\} = \Lambda(\{\Delta x_3^\mu\}) \qquad (5.3)$$

Parallel transport of $\Delta \mathbb{X}_1$ and $\Delta \mathbb{X}_2$ yields $\Delta \mathbb{X}_2 + \Delta \mathbb{X}_1 = \Delta \mathbb{X}_1 + \Delta \mathbb{X}_2 = \Delta \mathbb{X}_3$ (both spacetime triangles in the parallelogram in Figure 5.2 are closed). Triangles being closed in the LAB frame and in the ROCKET frame implies

5.6

$$\Delta x_3^\mu = \Delta x_1^\mu + \Delta x_2^\mu \;, \quad \mu = 0,1,2,3 \qquad \text{(LAB)} \qquad (5.4)$$

$$\Delta \bar{x}_3^\mu = \Delta \bar{x}_1^\mu + \Delta \bar{x}_2^\mu \;, \quad \mu = 0,1,2,3 \qquad \text{(ROCKET)} \qquad (5.5)$$

Apply Eqs. (5.1)-(5.3) to Eq.(5.5). One obtains

$$\Lambda(\{\underbrace{\Delta x_3^\mu}_{\Delta x_1^\mu + \Delta x_2^\mu}\}) = \Lambda(\{\Delta x_1^\mu\}) + \Lambda(\{\Delta x_2^\mu\}) \quad \leftarrow \text{Eq.(5.4)}$$

Using Eq. 5.4 on obtains

$$\Lambda(\{\Delta x_1^\mu + \Delta x_2^\mu\}) = \Lambda(\{\Delta x_1^\mu\}) + \Lambda(\{\Delta x_2^\mu\})$$

which means that Λ is a linear transformation the set of vector components. Applying this transformation to the vector components of $\Delta \mathbb{X}$, one arrives at the explicit linear relationship of Eq.(5.1) in terms of the matrix equation

$$\Delta \mathbb{X}: \{\Delta x^\mu\} \rightsquigarrow \{\Delta \bar{x}^\mu\} = \begin{bmatrix} \Delta\bar{x}^0 \\ \Delta\bar{x}^1 \\ \Delta\bar{x}^2 \\ \Delta\bar{x}^3 \end{bmatrix} = \begin{bmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{bmatrix} \begin{bmatrix} \Delta x^0 \\ \Delta x^1 \\ \Delta x^2 \\ \Delta x^3 \end{bmatrix}$$

or, using <u>matrix notation</u>

$$[\Delta \bar{x}] = \Lambda [\Delta x] \qquad (5.6)$$

III. EINSTEIN SUMMATION CONVENTION

In terms of <u>index notation</u> the matrix Eq.(5.6) assumes the explicit form

$$\Delta \bar{x}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu \Delta x^\nu \;, \quad \mu = 0,1,2,3$$

$$\equiv \Lambda^\mu{}_\nu \Delta x^\nu \qquad \text{(Einstein summation convention)}$$

In Einstein's summation convention the pairwise occurring <u>summation index</u> is called a <u>dummy index</u>. This is because it can

can be replaced by any other index symbol, e.g.

$$\Lambda^{\mu}{}_{\nu}\Delta x^{\nu} = \Lambda^{\mu}{}_{\alpha}\Delta x^{\alpha} \qquad \mu = 0,1,2,3$$

Dummy indices always come in <u>distinct pairs</u> <u>only</u>. For example, consider the double sum involving the matrix $[\eta_{\mu\nu}] = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$. For this matrix the double sum using Einstein convention is

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\eta_{\mu\nu}\Delta x^{\mu}\Delta x^{\nu} \equiv \eta_{\mu\nu}\Delta x^{\mu}\Delta x^{\nu}$$

The fact that dummy indeces <u>must</u> occur <u>only</u> in <u>distinct</u> pairs incorporates an automatic error correction code in multilinear and tensor algebra. For example, it would be incorrect to ever try to write an expression such as

$$A_{\mu\mu}\Delta x^{\mu}\Delta x^{\mu} \qquad \text{(BAD!)}.$$

The error in using the summation convention in this way lies in the ambiguity

$$A_{\mu\mu}\Delta x^{\mu}\Delta y^{\mu} = \begin{cases} \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} A_{\mu\nu}\Delta x^{\mu}\Delta y^{\nu} = \cdots + A_{01}\Delta x^{0}\Delta y^{1} + A_{10}\Delta x^{1}\Delta y^{0} + \cdots \\ \text{or} \\ \sum_{\nu=0}^{3}\sum_{\mu=0}^{3} A_{\nu\mu}\Delta x^{\mu}\Delta y^{\nu} = \cdots + A_{10}\Delta x^{0}\Delta y^{1} + A_{01}\Delta x^{1}\Delta y^{0} + \cdots \end{cases}$$

The ambiguity results in two entirely different double sums.

Thus: (1) In a double or higher order sum, violating the distinctness of one pair of summation indeces as compared to any other pair leads to disaster, always.

and (2) Violating the distinction between free indeces and dummy indeces leads to to disaster as well.

Conclusions: 1. The point events $\{\mathcal{X}\}$ of spacetime form a vector space

2. DEFINITION A

A set of four numbers form the components of a 4-vector whenever their transformation from one frame to another is governed by a Lorentz transformation.

Comment: There does exist an objective mental process for arriving at a concept. It has three stages and is illustrated by the formation of the above 4-vector concept.

(i) Consider the collection of quadruples of numbers, arrays of four numbers.

(ii) Identify them by focusing on their common commensurable properties, namely that they are quadruples and that each one has numbers for its entries. Pick some particular quadruple, the four spacetime coordinates of a particular event as measured in a particular inertial frame. That quadruple for this event will serve as a reference standard for the next stage.

(iii) Compare and contrast this reference quadruple with the others obtained from the other inertial frames by asking and aswering: which ones are related to the reference quadruple by means of a Lorentz transformation, and which ones, by contrast, are not? The second serves as a foil – a complement – to the first. The answer to this question separates the set of quadruples into two mutually exclusive and jointly exhaustive categories: those quadruples which are Lorentz related to the reference quadruple and those which are not. With stage (i) and (ii) as the prerequisite, identifying that first category – which differs strikingly from the second – is the key to the formation of the concept, to the organic whole comprised of its Lorentz related units. Because of this relation, one says that each one of them "fits well" with any of the other units.

This means that one has produced a well-formed structure, a mental reality-based integration of quadruples into an organic whole using the reference quadruple as a four-dimensional standard throughout. This product is the 4-vector represented by the quadruple of spacetime coordinates of the above particular event measured in the above particular frame.

To arrive at the concept of a 4-vector one's mind has to follow a sequence of three well-defined stages. In telegraphic form they are "entity", "identity", and "unit". Ayn Rand in her "Introduction to Objectivist Epistemology" shows that this 3-stage development is the essence of the formation of concepts in general, and that that formation process is essentially mathematical in nature.

DEFINITION B

"A vector is an element of a vector space."