LECTURE 6

I. Definition of a vector

II. Examples of vectors

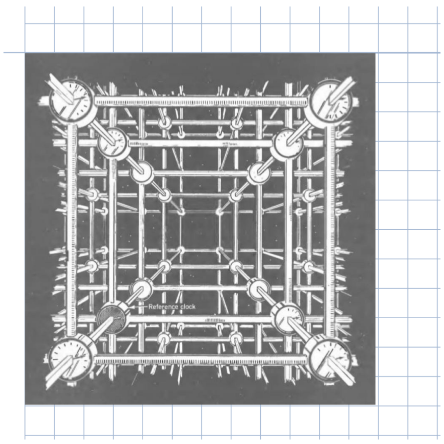
- a. Displacement 4-vectors
- b. 4-velocity
- c. 4-acceleration
- d. The propagation 4-vector

Read and chew

In MTW: Box 1.3; 2.4
Sect. 2.1-2.3

In T-W : Box 7.1

The mathematization of relativity physics is rooted in events, vectors, and higher order geometrical objects as perceived and measured by observers, each in his spacetime domain equipped with a lattice work of clocks and measuring rods.



Latticework of clocks and
rods for measuring events

I. Definition of a Vector

The set of four numbers

$$\{x^\nu : \nu = 0, 1, 2, 3\},$$

the attributes of an entity as measured by an observer in in his frame S, are said to be the components of a vector if

(i) the attributes of that entity in frame $\bar{S}$ yield

$$\{\bar{x}^\mu : \mu = 0, 1, 2, 3\}$$

and

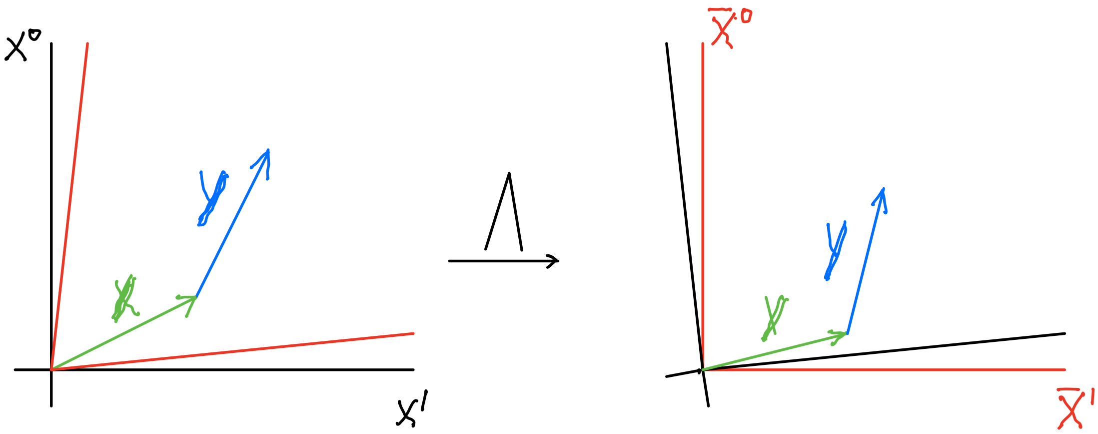
(ii) are related to those in S by the linear transformation

$$\bar{x}^\mu = \Lambda^\mu{}_\nu \, x^\nu \qquad \mu = 0, 1, 2, 3 \ .$$

In such a circumstance the entity is said to be a vector.

II. Examples of vectors.

a) Displacement vector



$$\overline{x^\mu + y^\mu} = \Lambda^\mu{}_\nu (x^\nu + y^\nu) = \Lambda^\mu{}_\nu x^\nu + \Lambda^\mu{}_\nu y^\nu$$

$$= \bar{x}^\mu + \bar{y}^\mu$$

$$\overline{cx^\mu} = \Lambda^\mu{}_\nu (c x^\nu) = c \Lambda^\mu{}_\nu x^\nu$$

$$= c \, \bar{x}^\mu$$

b) Four-velocity as the tangent to a curve

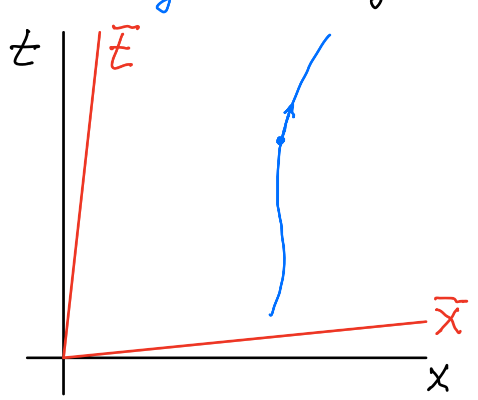


Figure 6.1: Worldline and the tangent at one of its events.

Its components are proportional,

$$\begin{bmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} \Delta\bar{t}\,\dfrac{1}{\sqrt{1-\beta^2}} + \Delta\bar{x}\,\dfrac{\beta}{\sqrt{1-\beta^2}} \\ \Delta\bar{t}\,\dfrac{\beta}{\sqrt{1-\beta^2}} + \Delta\bar{x}\,\dfrac{1}{\sqrt{1-\beta^2}} \\ \Delta\bar{y} \\ \Delta\bar{z} \end{bmatrix}$$

(i) Dividing by Δt yields

$$\frac{dx}{dt} = \frac{\frac{d\bar{x}}{d\bar{t}} + \beta}{1 + \frac{d\bar{x}}{d\bar{t}}\beta} \qquad \frac{dy}{dt} = \frac{\frac{d\bar{y}}{d\bar{t}}\sqrt{1-\beta^2}}{1 + \frac{d\bar{x}}{d\bar{t}}\beta}$$

This does not have the correct Lorentz transformation properties. In fact, it is epistemically stultifying. This is because it violates Rand's Razor, "Concepts are not to be multiplied beyond necessity" (whose corollary is "nor are they to be integrated in disregard of necessity")

(ii) To obtain a four-vector one must divide by an <u>invariant</u>,

$$\sqrt{(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2} = \Delta\tau = \sqrt{(\Delta\bar{t})^2 - (\Delta\bar{x})^2 - (\Delta\bar{y})^2 - (\Delta\bar{z})^2}$$

This leads one to consider

$$\left\{ \frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right\} \equiv \left\{ \frac{dx^\mu}{d\tau} : \mu = 0,1,2,3 \right\},$$

which are the components of a vector, the <u>four-velocity</u> (= tangent) of the curve

$$\mathcal{X}(\tau): \{ x^\mu(\tau) : t = x^0(\tau),\ x = x^1(\tau),\ y = x^2(\tau),\ z = x^3(\tau) \}, \qquad (6.1)$$

which here we consider to be parametrized by the proper time

$$\tau = \int_0^\tau d\tau = \int_0^\tau \sqrt{dt^2 - dx^2 - dy^2 - dz^2}$$

$$= \int_0^t \sqrt{1 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2}\, dt. \qquad (6.2)$$

Thus, once one knows the curve $x(t), y(t), z(t)$, and hence $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ relative to the frame coordinalized by t, x, y, and z, the integral, Eq. (6.2) yields the proper time τ which parametrizes the progress along the given worldline, Eq. (6.1)

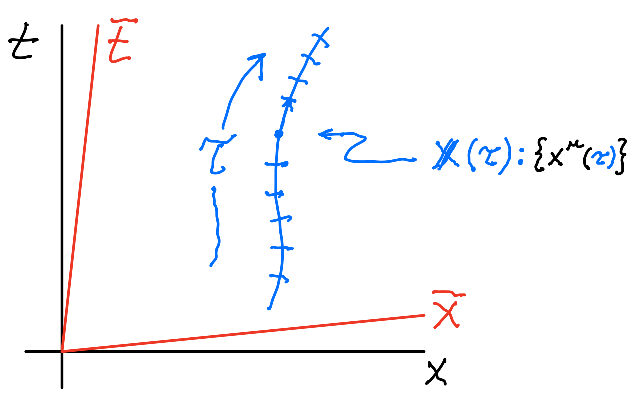


Figure 6.2: Worldline parametrized by its proper time.

(iii) The frame-independent parameter τ refers to the wrist-watch (a.k.a.

"proper") time of an observer comoving with the entity (e.g. an electron accelerated in an electro-magnetic field) executing the given (non-straight) worldline.

(iv) Depending on the context, one refers to the four-velocity

(a) in terms of its coordinates

$$\left\{ \frac{dx^\mu}{d\tau} \right\} \equiv \{ v^\mu(\tau) \} \quad (\text{"coordinate representation"})$$

or (b) more abstractly as the geometrical object

$$\frac{d\mathbb{X}(\tau)}{dt} \equiv \mathbf{v}(\tau),$$

which suppresses explicit reference to any coordinate components, which are <u>always</u> implied. Thus a vector must have <u>some</u> but may have <u>any</u>!

(v) The four-velocity is a <u>vector</u> because its coordinate components obey the Lorentz transformation law between one frame and another

$$\begin{bmatrix} \frac{dt}{d\tau} \\ \frac{dx}{d\tau} \\ \frac{dy}{d\tau} \\ \frac{dz}{d\tau} \end{bmatrix} = \begin{bmatrix} ch\theta & sh\theta & 0 & 0 \\ sh\theta & ch\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{d\bar{t}}{d\tau} \\ \frac{d\bar{x}}{d\tau} \\ \frac{d\bar{y}}{d\tau} \\ \frac{d\bar{z}}{d\tau} \end{bmatrix}$$

i.e. $$\left[\frac{dX}{d\tau}\right]_{S} = \Lambda \left[\frac{dX}{d\tau}\right]_{\bar{S}}$$ ("MATRIX NOTATION")

$$\frac{dx^{\mu}}{d\tau} = \Lambda^{\mu}{}_{\nu} \frac{dx^{\bar{\nu}}}{d\tau}$$ ("INDEX NOTATION")

c) One can readily verify that,

$$\left(\frac{d^2t}{d\tau^2}, \frac{d^2x}{d\tau^2}, \frac{d^2y}{d\tau^2}, \frac{d^2z}{d\tau^2}\right) \equiv \left\{\frac{d^2x^{\mu}}{d\tau^2}\right\},$$

the components of the four-acceleration are also the components of a four-vector.

d) A very important example of another four-vector is the propagation four-vector. It arises in the context of a wave function, whose form in a particular reference frame is typically

$$\psi = \cos\left(k_x x + k_y y + k_z z - \omega t\right),$$

a solution to the wave equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{\partial^2\psi}{\partial t^2} = 0.$$

1. PHASE IN A GIVEN FRAME OF REFERENCE

The key to understanding the behavior of this wave function is its phase function

$$\boxed{\varphi(x,y,z,t) = k_x x + k_y y + k_z z - \omega t}$$

This phase function mathematizes the location of the crests of the wave where $\cos\varphi = 1$.

6,8

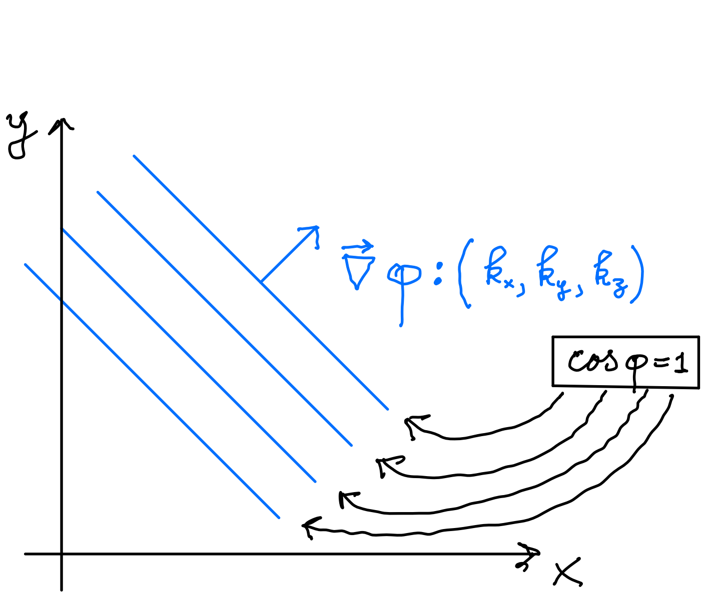
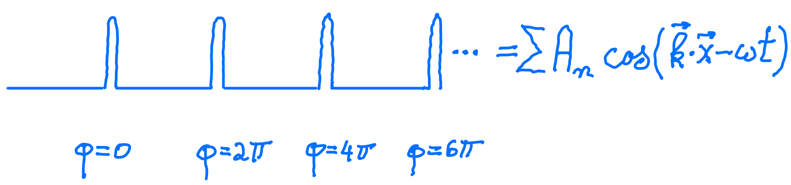


Figure 6.3: Phase isograms of wave pattern at t = fixed

Q: What is the wave pattern and the wave frequency in another frame?

A: Don't discuss the wave amplitude because its transformation properties are in general complicated. Instead, focus on a train of pulses, which is a Fourier superposition of wave functions.



$$\boxed{\begin{aligned}\varphi &= 2\pi\cdot(\#\text{ of pulses})\\ &= \text{"phase"}\end{aligned}}$$

The spacetime behaviour of the wave function is controlled by the following four phase parameters:

ω = increase in phase per unit time (= "frequency")

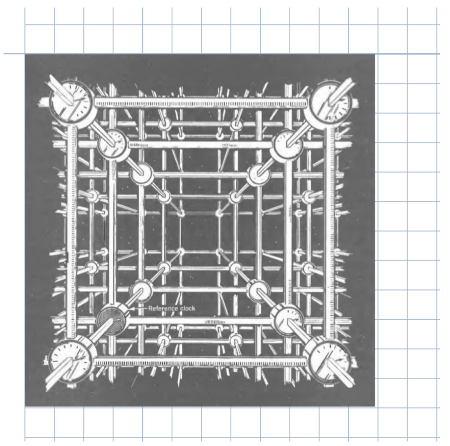
k_x = increase in phase per unit x $\left(= \text{"x-wave \# into x-direction"}\right)$

k_y = increase in phase per unit y $\left(= \text{"y-wave \# into y-direction"}\right)$

k_y = increase in phase per unit y $\left(= \text{"y-wave \# into y-direction"}\right)$

2. PHASE IN DIFFERENT FRAMES OF REFERENCE

The phase φ of a particular pulse is an <u>invariant</u>, the same in all inertial reference frames S, $\bar{S}$, $\bar{\bar{S}}$, $\cdots$. This is because of the Principle of Relativity. Indeed, when a particular pulse travels through reference frame S like the one pictured,



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and triggers an event, then this event, and the pulse maximum causing it, is also recorded in frame $\bar{S}$. The property of a particular maximum of a wave disturbance triggering a specific event, when observed in one frame, is also observed to be triggered by a wave maximum relative to any other frame. This manifestation of the Principle of Relativity is mathematized by the statement that the phase of the wave is an invariant

$$k_x x + k_y y + k_z z - \omega t = \bar{k}_x \bar{x} + \bar{k}_y \bar{y} + \bar{k}_z \bar{z} - \bar{\omega} \bar{t}. \qquad (6.3)$$

Insert

$$\begin{aligned} t &= \bar{t}\, ch\theta + \bar{x}\, sh\theta \\ x &= \bar{t}\, sh\theta + \bar{x}\, ch\theta \\ y &= \bar{y} \\ z &= \bar{z} \end{aligned} \qquad (6.4)$$

into this equation. Collect terms. Equation (6.3) holds for all $\bar{x}, \bar{y}, \bar{z}$, and $\bar{t}$. Consequently,

$$\begin{aligned} \bar{\omega} &= \omega \underbrace{ch\theta}_{\frac{1}{\sqrt{1-\beta^2}}} - k_x \underbrace{sh\theta}_{\frac{\beta}{\sqrt{1-\beta^2}}} \\ \bar{k}_x &= k_x\, ch\theta - \omega\, sh\theta \\ \bar{k}_y &= k_y \\ \bar{k}_z &= k_z \quad . \end{aligned} \qquad (6.5)$$

Thus $(\omega, k_x, k_y, k_z) \equiv (k^0, k^1, k^2, k^3) = \{k^\mu\}$ are the components of the wave <u>propagation four-vector</u>.

COMMENT:

The wave function ψ satisfies the wave equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial^2 \psi}{\partial t^2} = 0.$$

Consequently, the wave propagation four-vector satisfies the following "dispersion" relation

$$\boxed{-\omega^2 + k_x^2 + k_y^2 + k_z^2 = 0}\,.$$

6.11

This is again an invariant, i.e.

$$\begin{bmatrix} k^0 & k^1 & k^2 & k^3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k^0 \\ k^1 \\ k^2 \\ k^3 \end{bmatrix} = \begin{bmatrix} \bar{k}^0 & \bar{k}^1 & \bar{k}^2 & \bar{k}^3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{k}^0 \\ \bar{k}^1 \\ \bar{k}^2 \\ \bar{k}^3 \end{bmatrix}$$

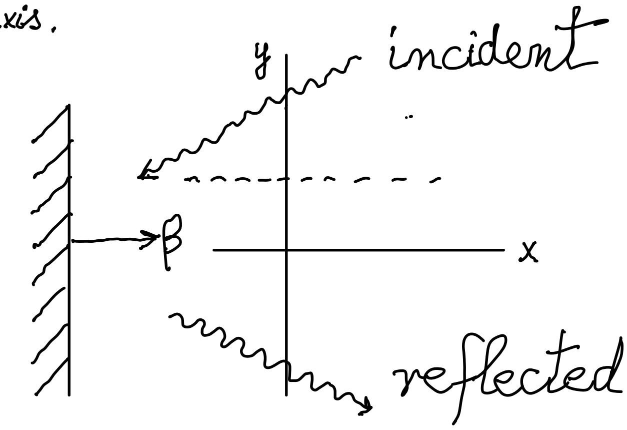
or

$$k^\mu \eta_{\mu\nu} k^\nu = \bar{k}^\mu \eta_{\mu\nu} \bar{k}^\nu = 0$$

Thus, the wave propagation four-vector is a <u>null vector</u> and it is a null vector in <u>every</u> reference frame.

Problem A: $\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$ is a frame invariant. Determine its value by means of a mathematical calculation.

Problem B: Consider a mirror in the lab moving with speed $\beta = \frac{v}{c}$ along the lab's x-axis.



A laser pulse of frequency ω and given incident propagation direction impinges on the moving mirror.

6.12

Determine the reflected frequency and the components of the reflected 3-d pulse propagation vector $\vec{k}_{ref}$ in the Lab frame.