# LECTURE 8



Accelerated frame via inertial frame invariats

I. Instantaneous inertial frame invariants

II. Spacetime geometry of a uniformly linearly accelerated particle.

In MTW read Chapter 6,
in particular Sections 6.1, 6.2, and 6.3

# I. Lorentz Frame Invariants

a) Given a particle and its history

$$\mathbb{X}(\tau): \quad \{t(\tau), x(\tau), y(\tau), z(\tau)\},$$

i.e. its world line parametrized by its proper time

$$\tau = \int_0^\tau d\tau = \int \sqrt{(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2}$$

$$= \int^{t(\tau)} \sqrt{1 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2}\, dt; \quad \frac{dt}{d\tau} = \frac{1}{\sqrt{\phantom{x}}}$$

its 4-velocity

$$\frac{d\mathbb{X}}{d\tau} \equiv \mathbb{U} : \left\{\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right\} = \frac{dt}{d\tau} \times \left\{1, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\} \equiv \left\{\frac{dx^\mu}{d\tau} ; \mu = 0,1,2,3\right\} \equiv \{U^\mu ; \mu = 0,1,2,3\}$$

has components, $u^\mu = \frac{dx^\mu}{d\tau}$, $\mu = 0,1,2,3$, have values that depend on the Lorentz frame in which the particle has been observed. Relative to a different frame, say $\bar{S}$, that same 4-velocity has different components:

$$\frac{d\mathbb{X}}{d\tau} \equiv \mathbb{U} : \left\{\frac{dx^{\bar{0}}}{d\tau}, \frac{dx^{\bar{1}}}{d\tau}, \frac{dx^{\bar{2}}}{d\tau}, \frac{dx^{\bar{3}}}{d\tau}\right\} \equiv \left\{\frac{dx^{\bar{\mu}}}{d\tau} ; \bar{\mu} = 0,1,2,3\right\} \equiv \{U^{\bar{\mu}} ; \bar{\mu} = 0,1,2,3\}.$$

The overline over the coordinate labels serves as a reminder that the coordinate compenents $u^{\bar{0}}, u^{\bar{1}}, u^{\bar{2}}, u^{\bar{3}}$ are relative the Lorentz frame $\bar{S}$.

In light of the invariance of the (squared) interval,

$$(\Delta\tau)^2 = (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2 \equiv -\eta_{\mu\nu}\,\Delta x^\mu \Delta x^\nu,$$

namely,

$$-\eta_{\mu\nu}\frac{\Delta x^\mu}{\Delta\tau}\frac{\Delta x^\nu}{\Delta\tau} = -\eta_{\bar{\mu}\bar{\nu}}\frac{\Delta x^{\bar{\mu}}}{\Delta\tau}\frac{\Delta x^{\bar{\nu}}}{\Delta\tau} = 1$$

where $\eta_{\mu\nu}$ are the components of the matrix

$$[\eta_{\mu\nu}] = [\eta_{\bar{\mu}\bar{\nu}}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

one infers that

$$\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=\eta_{\bar\mu\bar\nu}\frac{dx^{\bar\mu}}{d\tau}\frac{dx^{\bar\nu}}{d\tau}=-1$$

or

$$\eta_{\mu\nu}u^\mu u^\nu=\eta_{\bar\mu\bar\nu}u^{\bar\mu}u^{\bar\nu}=-1$$

is an invariant whose value is $-1$. Because it is the same in in all Lorentz frames, for the sake of "unit-economy" (to prevent perceptual overload) one writes these two equation simply as

$$\boxed{\mathbf{u}\cdot\mathbf{u}=-1}\,.$$

This equation states in highly condensed form that $\mathbf{u}\cdot\mathbf{u}$ is an invariant and that its value is $-1$.

b) Analogous statements hold for any other 4-vector $\mathbf{v}$ whose invariant is

$$\eta_{\mu\nu}v^\mu v^\nu=\eta_{\bar\mu\bar\nu}v^{\bar\mu}v^{\bar\nu}\equiv\mathbf{v}\cdot\mathbf{v}$$

c) From physical considerations (Lectures 5) we know that Lorentz spacetime is a vector space. Thus, in light of Definition A in Lecture 4, $\mathbf{u}+\mathbf{v}$ is also a vector. Its invariant is

$$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})=\underbrace{\mathbf{u}\cdot\mathbf{u}}_{\text{"invariant"}}+\underbrace{\mathbf{v}\cdot\mathbf{v}}_{\text{"invariant"}}+\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}=\text{"invariant"}$$

Consequently $\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}$ is also an invariant. Furthermore,

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{u}\equiv\eta_{\mu\nu}v^\mu u^\nu &= \eta_{\nu'\mu'}v^{\nu'}u^{\mu'} && \text{(change summation indeces: } \mu=\nu';\ \nu=\mu')\\
&=\eta_{\mu'\nu'}v^{\nu'}u^{\mu'} && ([\eta_{\nu'\mu'}]\text{ is a symmetric matrix})\\
&=\eta_{\mu\nu}v^\nu u^\mu && \text{(drop the "primes")}\\
&=\eta_{\mu\nu}u^\mu v^\nu && \text{(multiplication is commutative)}\\
&=\mathbf{u}\cdot\mathbf{v} && \text{(definition of } \mathbf{u}\cdot\mathbf{v})
\end{aligned}$$

Thus

$$u\cdot v + v\cdot u = 2\,u\cdot v$$

is also an invariant:

$$\boxed{u\cdot v = \text{"invariant"}}$$

Comment:

The above line of reasoning from $u\cdot v$ to $v\cdot u$ is referred to informally as "index gymnastics." Furthermore, the real number $u\cdot v$ constructed from $u$ and $v$ is the spacetime inner product for Lorentz spacetime. This is a concept we need to return to later in developing the multilinear algebra of tensors.

d) The invariant $u\cdot v$ is a mathematical projection. It mathematizes observational relations between different Lorentz frames.

Consider the emission in the Lab frame a radiation pulse with propagation 4-vector

$$\mathbf{k} : \{k^0, k^1, k^2, k^3\} \qquad (8.1)$$

relative to the Lab frame.

Determine the frequency $\bar{\omega}$ of $\mathbf{k}$ as measured in the rocket frame $\bar{S}$ whose 4-velocity is

$$\bar{\mathbf{u}} : (\bar{u}^0, \bar{u}^1, \bar{u}^2, \bar{u}^3) \qquad (8.2)$$

as measured in the Lab frame $S$.

An efficient way of solving this problem is to take advantage of the physical meaning of the invariant

$$\bar{u}\cdot\mathbf{k} = -\bar{u}^0 k^0 + \bar{u}^1 k^1 + \bar{u}^2 k^2 + \bar{u}^3 k^3 = -\bar{u}^0 k^0 + \bar{u}^1 k^1 + \bar{u}^2 k^2 + \bar{u}^3 k^3 \qquad (8.3)$$

It is known that the 4-velocity $u$ has zero spatial components in its own comoving frame:

$$\bar{u}: (\bar{u}^0, \bar{u}^1, \bar{u}^2, \bar{u}^3) = (1, 0, 0, 0)$$

We also know that $\bar{k}^0 = \bar{\omega}$ is the zeroeth component of $\mathbf{k}$ as measured in the Rocket frame:

$$\mathbf{k}: (\bar{k}^0, \bar{k}^1, \bar{k}^2, \bar{k}^3) = (\bar{\omega}, \bar{k}_x, \bar{k}_y, \bar{k}_y)$$

Consequently,

$$\bar{u} \cdot \mathbf{k} = -\bar{\omega}$$

On the other hand, using Eqs. (8.1) and (8.2), the Lab observer readily calculates the r. h. side of Eq. (8.3) on page 8.3,

$$\bar{u} \cdot \mathbf{k} = -\bar{u}^0 k^0 + \bar{u}^1 k^1 + \bar{u}^2 k^2 + \bar{u}^3 k^3 .$$

Consequently, based on his own data only, the Lab observer S predicts that frequency of $\mathbf{k}$ as observed by $\bar{S}$ is

$$\boxed{\bar{\omega} = \bar{u}^0 k^0 - \bar{u}^1 k^1 - \bar{u}^2 k^2 - \bar{u}^3 k^3}$$

$$\boxed{\bar{\omega} = -\mathbf{u} \cdot \mathbf{k}}$$

II. Einstein in 1907 introduced two innovations, one in physics the other in mathematics.

(i) He opened new vistas in physics by extending his analysis of spacetime from inertial to accelerated frames of reference, all within the perview of Special Relativity.

(ii) He introduced a new method into mathematics: The method of instantaneous inertial frames, which after W.W.II. have been called the tangent spaces of a manifold.

This two-part innovation is stated by saying that an <u>accelerated frame is a 1-parameter family of inertial frames.</u>

The archetypical accelerated frame is one that arises in physics, namely the one where the comoving observer experiences a <u>constant</u> <u>linear</u> acceleration as measured by his accelerometer.

Based on this physical condition, the problem to determine the global spacetime trajectory of the spatial origin, $x^1 = x^2 = x^3 = 0$, of this frame.

This determination is a two step process:

(i) Set up a system of appropriate differential equations,

(ii) Determine their solution.

But first with the acceleration non-zero, establish the necessary condition between the trajectory's 4-velocity and the 4-acceleration.

Let $\mathbf{u} = \frac{d\mathbf{x}}{d\tau} : u^\mu = \frac{dx^\mu}{d\tau}$ be the 4-velocity of the world line

Let $\mathbf{a} = \frac{d\mathbf{u}}{d\tau} : a^\mu = \frac{du^\mu}{d\tau}$ be the 4-acceleration of the worldline

where $\boxed{\mathbf{u}\cdot\mathbf{u} = u^\mu u^\nu \eta_{\mu\nu} = -1}$

implies

$$0 = \frac{d}{d\tau}(-1) = \frac{d}{d\tau}\left(u^\mu u^\nu \eta_{\mu\nu}\right) = a^\mu u^\nu \eta_{\mu\nu} + u^\mu a^\nu \eta_{\mu\nu}$$

$$= 2a^\mu u^\nu \eta_{\mu\nu} \qquad \text{(after some index "gymnastics)}$$

Thus

$$\boxed{\mathbf{a}\cdot\mathbf{u} = 0},$$

which says that the acceleration 4-vector is <u>always</u> Lorentz orthogonal to the 4-velocity.

Comment:

The above boxed equation expresses the fact that $\mathbf{a}$ is a space-like vector. Indeed, relative to the co-accelerating frame where the spatial components vanish on has

$$0 = \mathbf{a}\cdot\mathbf{u} = -a^{\bar{0}}\cdot 1 + a^{\bar{1}}\cdot 0 + a^{\bar{2}}\cdot 0 + a^{\bar{3}}\cdot 0.$$

Thus, $\mathbf{a} : (0, a^{\bar{1}}, a^{\bar{2}}, a^{\bar{3}})$

is a purely space-like vector and

$$\boxed{\mathbf{a}\cdot\mathbf{a} = (a^{\bar{1}})^2 + (a^{\bar{2}})^2 + (a^{\bar{3}})^2 \equiv g^2} = \left(\begin{array}{l}\text{acceleration}\\ \text{measured by a}\\ \text{comoving}\\ \text{spring balance}\end{array}\right)^2$$

This is the squared magnitude of the acc'n as recorded in the coaccelerating frame.

b) Set up the governing differential and algebraic equations that determine the shape of the world line:

| | | |
|---|---|---|
| $\frac{dx^0}{d\tau} = u^0$ | $\frac{du^0}{d\tau} = a^0$ | $(u^0)^2 - (u^1)^2 = 1 \quad (*)$ |
| $\frac{dx^1}{d\tau} = u^1$ | $\frac{du^1}{d\tau} = a^1$ | $a^0u^0 - a^1u^1 = 0 \quad (**)$ <br> $-(a^0)^2 + (a^1)^2 = g^2 \quad (***)$ |

c) Solve this system by first using the algebraic equations to reduce the number of variables. Do this by expressing $a^0$ and $a^1$ in terms of $u^1$ and $u^0$. The result is

$$\boxed{a^1 = g u^0 \quad \text{and} \quad a^0 = g u^1}$$

Thus the system of differential equation reduces to

| | |
|---|---|
| $\frac{dx^0}{d\tau} = u^0$ | $\frac{du^0}{d\tau} = g u^1$ |
| $\frac{dx^1}{d\tau} = u^1$ | $\frac{du^1}{d\tau} = g u^0$ |

Focusing on a trajectory with <u>constant</u> $g$, one finds that the velocity equation becomes

$$\frac{d^2u^0}{d\tau^2} - g^2u^0 = 0$$

It solution is

$$u^0 = A\,\mathrm{ch}\,g\tau + B\,\mathrm{sh}\,g\tau$$

and

$$u^1 = \frac{1}{g}\frac{du^0}{d\tau} = A\,\mathrm{sh}\,g\tau + B\,\mathrm{ch}\,g\tau$$

d) A differential equation is never solved until the initial and/or boundary conditions are imposed on the general solution. These conditions mathematize an actual physical process.

We impose initial conditions and one constraint condition:

(i) At $\tau=0$ $u^1=0$ (start at rest): $\frac{dx^1}{d\tau}=u^1=0$ $\Rightarrow B=0$

(ii) Unit 4-velocity constraint $(u^0)^2-(u^1)^2=1$ $\Rightarrow A=0$

The coordinates of the spacetime trajectory are

$$x^0(\tau)=\frac{1}{g}\,\mathrm{sh}\,g\tau+c_0$$

$$x^1(\tau)=\frac{1}{g}\,\mathrm{ch}\,g\tau+c_1$$

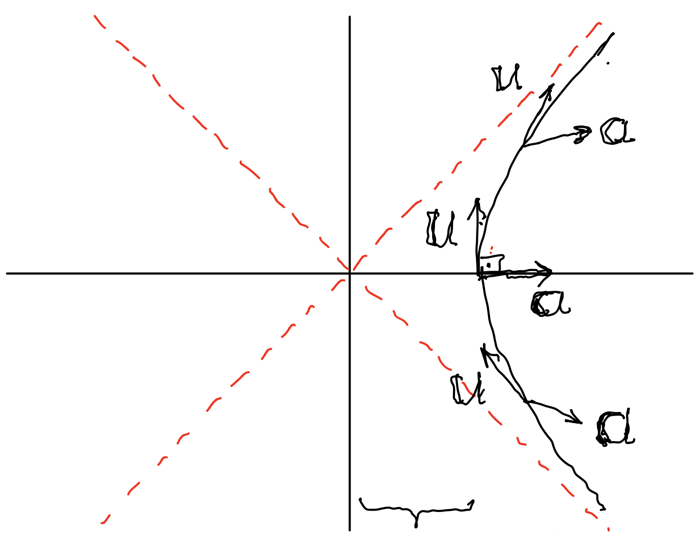
(iii) At $\tau=0$ $x^0=0$ } initial event on the trajectory } $\Longrightarrow$ $c_0=0$

(iv) At $\tau=0$ $x^1=\frac{1}{g}$ } $\Longrightarrow$ $c_1=\frac{1}{g}$

The equation for the spacetime trajetory is therefore

$$(x^1)^2-(x^0)^2=\frac{1}{g}\,,$$

a hyperbolic world line.




$g^{-1}$ = Fermi-Walker ("Headstart") distance

Figure 8.1: Spacetime trajectory of a particle accelerated linearly with constant acceleration $g = g_{conv.}/c^2$. The wordline of such a particle is a spacetime hyperbola.

As an example consider an acceleration

$$g_{conventional} = 1000 \left[\frac{cm}{sec^2}\right]$$

Consequently

$$\frac{d^2x}{d(c\,t_{conv.})^2} = g = \frac{g_{conv}}{c^2} = \frac{10^3}{10^{21}} = 10^{-18}\left[\frac{1}{cm}\right]$$

Thus

$$\frac{1}{g} = 10^{18}\,cm \;[\approx 1 \text{ light year}]$$

The accelerated particle/rocket can outrace a photon if given a headstart of ~1 l. year.