LECTURE 1

The concept "Gravitation": Where does it come from?

A. Gravitation's observational basis

B. Mathematization of gravitation according to

I. Galileo

II. Kepler

III. Newton

IV. Lagrange & Hamilton

V. Einstein

The theme of Math 5757 is to grasp the nature of the world, in particular, the existence and the nature of the concept of "gravitation", which is the subject in this theme.

A.) To develop this, our thinking must start with information received from the world.

In the case of gravitation this information is in the specific form of the laws of motion of bodies. It is precisely in terms of the observed motion of bodies that one arrives at the concept "gravitation."

More generally, the concept "gravition" is the (mental) product of the integration of a constellation of concepts each one which is formed by a process of "measurement omission". \footnote{ The process of concept formation is a process of "measurement omission". It is explained on pages 11–18 of chapter 2 ("Concept Formation") in "Introduction to Objectivist Epistemology" by Ayn Rand; also summarized in the Q&A on pages 137–139.}

The formation of the concept "gravitation" starts out by observing various kinds of motions of bodies and then singling out a particular type which is different from all the rest, e.g., the motion of the moon, or a falling apple as compared to pushing a ball makes it move. The common features which particularizes that type of motions are the characteristic imprints, the signature that gravitation imparts to the motion of bodies.

B.) Historically, what nowadays is identified as the gravitation phenomenon has been the chief motivating force for the mathematization (= "mathematical formulation") of the laws of motion. This is because they are the premier tool for identifying what gravitation is. It was Galileo, Kepler, Newton, Einstein and others who, each in their own way mathematizes gravition in their own way.

I. $\boxed{\text{Galileo:}}$ ("E pur si moove" $\longrightarrow$ "And yet it moves")

"Independence of Horizontal and Vertical Motion".

Using <u>the experimental method</u> in studying the motion of a projectile, Galileo found its horizontal and its vertical motion are independent of each other.

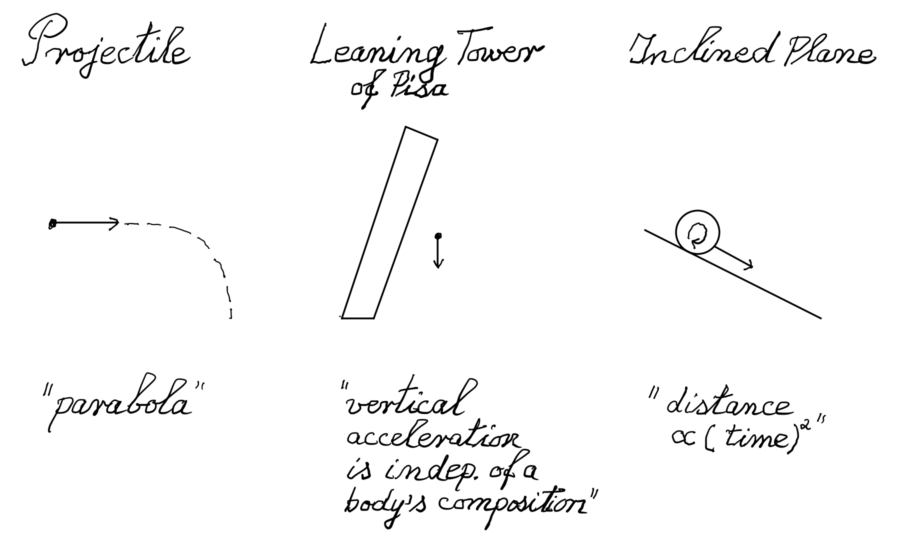


Figure 1.1: Galileo's mathematizations of the motion of bodies.

The independence of these motions are mathematized by the two statements:

1. Horizontal velocity = const.

1.3

2. Vertical acceleration = const.

a) Vertical velocity = $g \times$ time

b) Vertical distance = $\frac{1}{2} g \cdot (\text{time})^2$

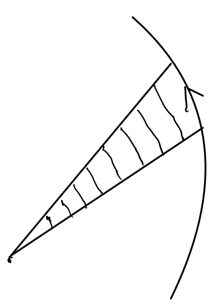
Comment 1.1

1. The fact that the horizontal velocity is constant is a special case of the law of inertial motion of bodies ("Newton's 1st law of motion").

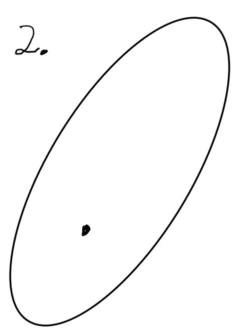
2. Although the concept <u>gravitation</u> was not yet known to Galileo, he identified its <u>imprint</u> on the motion of bodies by the proportionality constant g between the vertical velocity and the time of travel of the body (= point particle).

II. $\boxed{\text{Kepler:}}$ "Kepler's 3 laws of planetary motion."

From Tycho Brahe's direct observations Kepler, by inductive reasoning ("from the particular to the general"), mathematized the motion of planets (and moons) into his three laws.

1. 

Areal velocity = constant

2. 

Elliptic orbit

3. $GM^1 = \omega^2 R^3$

1-2-3 law

Figure 1.2: Kepler's 3 laws of planetary motion. The constant G in his 1-2-3 law is Newton's gravitational constant, $G = \frac{1}{15{,}000} \frac{m^3}{(kg)(sec)^2} = \frac{1}{15{,}000{,}000} \frac{cm^3}{(g)(sec)^2}$.

1. The radius vector sun-planet sweeps out equal areas in equal times

2. Planets travel in ellipses, with the sun at one of their foci.
3. $G \times (\text{Mass of the Sun}) = \left(\frac{2\pi}{\text{period}}\right)^2 \times (\text{major axis})^3$ is Kepler's "1-2-3 Law", namely

$$GM' = \omega^2 R^3 \qquad (1.1)$$

Comment: Without Kepler's Herculean work of extracting his 3 mathematical laws from Tycho Brahe's astronomical observations, it is doubtful that Newton could have formulated his law of universal gravitation in such short order. (See Lecture 0 for how he obtained his law from Kepler's 3 laws.)

III. $\boxed{\text{Newton}}$ "Universal Law of Gravitation."

(Galileo's motion of bodies) + (Local vector calculus) + $\left(\vec{F} = m\frac{d^2\vec{x}}{dt^2}\right)$ + (Kepler's 3 laws of planetary motion) + (observational data about planets, comets, moons, cannon balls, apples, etc.)

implies

$$m_{inertial}\,\frac{d^2\vec{x}}{dt^2} = -m_{grav'l}\,\frac{GM}{r^3}\,\vec{x} \qquad (1.2)$$

Exercise (Newton's law of gravitation)

Show that

$$\{\text{Kepler's 3 laws}\} \underset{b)\Leftarrow}{\overset{a)\Rightarrow}{\rightleftarrows}} (\text{grav'l force field}) = \frac{GM}{r^3}\,\vec{x}$$

Comment 1.2

Showing the implication "⇒" entails only differential calculus (See P 5-10 of Lecture 0 \footnote{ Being endemic to a cultured physicist or mathematician, this is also done in OSU's Math 1181H and 4551.}.)

Comment 1.3

Showing the implication "⇐" is more challenging mathematically. This is because it entails integral calculus.

Comment 1.4

By introducing his 2^{nd} Law,

$$\frac{d}{dt}\left(m\frac{d\vec{x}}{dt}\right) = \vec{F},$$

Newton accomplished several feats in one fell swoop:

(i) He introduced a new concept, the inertial mass of a body.

(ii) Whereas Kepler and Galileo mathematized motion in terms of global geometrical figures (ellipses, parabolas, etc.), Newton, having introduced the concept of mass and $F = ma$, did so in terms of locally defined differential equations such as the boxed Eq. (1.2) on page 1.4, or more generally

$$\boxed{m_{\text{inertial}} \frac{d^2\vec{X}}{dt^2} = -m_{\text{grav'l}}\, G \sum_i M_i \frac{\vec{X}-\vec{X}_i(t)|}{|\vec{X}-\vec{X}_i(t)|^3}} \qquad (1.3)$$

whenever there are several gravitation forces due to masses M_1 at $\vec{x}_1(t)$, M_2 at $\vec{x}_2(t)$, ….

(iii) Newton gave a local definition of acceleration by means of a <u>double limiting</u> process applied to differential equations.

Comment 1.5

Q: What is the difference between Newton's contribution to our understanding of the motion of bodies and that due to Kepler and Galileo?

A: Galileo and Kepler's geometrical figures mathematize the motion of bodies <u>kinematically</u>, i.e. <u>without</u> any references to their masses. By contrast, Newton's equations for the motion of bodies mathematize them <u>dynamically</u> <u>in terms</u> of their <u>masses</u>.

IV. Euler, Lagrange, Hamilton

The time interval between Newton and Einstein (17^{th}, 18^{th}, and 19^{th} century) was marked by the development of the <u>"Hamilton's Principle"</u> of least action by Euler, Lagrange, and Hamilton. This principle uses the calculus of variations to replace the task of setting up Newton's vectorial equations of motion with the much easier task of extremizing a scalar integral, the "action" of the mechanical system,

$$\int (K.E. - P.E.)\,dt = \text{extremum!} \qquad (1.4)$$

with the implication that

$$\delta\left\{\int (K.E. - P.E.)\,dt\right\} = 0 \qquad (1.5)$$

The main virtue of this formulation of the classical laws of motion is that the action of a mechanical system (i) is a scalar and (ii) that the extremum of this scalar is <u>independent of</u>

the choice of coordinates used to describe the mechanical system. If one reexpresses the Lagrangian K.E.−P.E. in terms of different coordinates, then the resulting Euler-Lagrange equations of motion (whose solution extremizes the action) still describes the same mechanical system, but relative to the new set of coordinates.

Q1: What is the physical origin of Hamilton's Principle as formulated by Lagrange?

A1: The observation-based reasoning leading to this principle is given on P1-7 of the attached Appendix "Lagrangian Mechanics and the 3-Body Problem.

Q2: Is it possible to give a non-trivial application of this principle?

A2: Yes. Pages 9-25 in the attached Appendix develop the theory of the (i) "Restricted 3-Body Problem" and (ii) the motion a charged particle in the crossed electric-magnetic field of a magnetron, which is used in radar and microwave ovens.

V. Einstein

By the time Einstein started examining the concept "gravitation", he had at his disposal, and then made excellent use of, the highly developed art of analytical mechanics as formulated by Lagrange and Hamilton.

In 1913 he took a key step. Using Hamilton's principle of least action, he equated Hamilton's action integral to the coordinate frame independent length

$$\int d\tau = \int \sqrt{-g_{\mu\nu}(x)\, dx^\mu dx^\nu} \qquad (1.6)$$

of a worldline between two events, and then pointed out that the metric tensor-field

$$g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu \qquad (1.7)$$

is where gravitation stamps its imprints, i.e. characterizes the gravitational field.

Thus instead of using Eq. (1.2) on page 1.4 to mathematize gravitation's imprints on the motion of bodies, Einstein applied Hamilton's variational action principle, Eq. (1.5) on page 1.5, to

$$\delta\left\{\int\sqrt{-g_{\mu\nu}(x)\,dx^\mu dx^\nu}\right\}=0\,. \qquad (1.8)$$

He thus obtained

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}=0\,, \quad \mu=0,1,2,3. \qquad (1.9)$$

These, we shall see,

(a) reduce to Newton's Eq. (1.2) on page 1.4,

$$m_{inertial}\frac{d^2\vec{x}}{d(ct)^2}+m_{grav'l}\vec{\nabla}\frac{\phi_{grav'l}(\vec{x},t)}{c^2}=0\,, \qquad (1.10)$$

in terms of the Newtonian gravitational potential $\phi_{grav'l}(\vec{x},t)$ \footnote{The units of $\phi_{grav'l}$ are energy/mass, so that $\frac{\phi_{grav'l}}{c^2}$ is dimensionless.} and

(b) express the imprints of gravitation on the motion of bodies in the form of geodesic states of motion on a spacetime manifold with a metric tensor field, and hence

(c) mathematizes gravitation in geometrical terms.