# Appendix to Lecture 10 and 26

The vector(ial) measure of an as-yet-to-be specified area is

$$e_\ell d^2\underset{\sim}{\Sigma}^\ell \equiv d^2\vec{\underset{\sim}{\Sigma}} \equiv \overset{(2)}{\vec{\underset{\sim}{\Sigma}}} = e_\ell\, \epsilon^\ell{}_{ij}\, \frac{dx^i\wedge dx^j}{2!};$$

We have 1) $e_\ell\, \epsilon^\ell{}_{ij}\, dx^i\wedge dx^j/2!\,(\vec u,\vec v) \equiv \vec u\times\vec v$

and 2.) $d\left(e_\ell d^2\Sigma^\ell\right) = 0$

PROOF:

$$d\left(e_\ell d^2\underset{\sim}{\Sigma}^\ell\right) = d\left(e_\ell\, \epsilon^\ell{}_{ij}\frac{dx^i\wedge dx^j}{2!}\right) = d\left(e_\ell\, g^{\ell k}\sqrt{g}\,[kij]\frac{dx^i\wedge dx^j}{2!}\right)$$

$$= \left[de_\ell\; g^{\ell k} + e_\ell\, dg^{\ell k} + e_\ell\, g^{\ell k}\frac{d\sqrt{g}}{\sqrt{g}}\right]\sqrt{g}\,[kij]\wedge\frac{dx^i\wedge dx^j}{2!}$$

$$= \Big[\underbrace{e_n\Gamma^n{}_{\ell r}dx^r g^{\ell k}}_{1} + \underbrace{e_\ell(-)g^{\ell r}g^{sk}dg_{rs}}_{2} + \underbrace{e_\ell\, g^{\ell k}\frac{d\sqrt{g}}{\sqrt{g}}}_{3}\Big]\sqrt{g}\,[kij]\frac{dx^i\wedge dx^j}{2!}$$

Recall that

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^r}\left(\sqrt{g}\,u^r\right) = u^r{}_{;r} = u^r{}_{,r} + u^s\Gamma^r{}_{sr}$$

$$u^r{}_{,r} + u^r\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^r} = u^r{}_{,r} + u^r\Gamma^s{}_{rs} \Rightarrow \boxed{\Gamma^s{}_{rs} = \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^r} = \frac{1}{2}g^{sm}\left(\cancel{g_{mr,s}} + g_{ms,r} - \cancel{g_{rs,m}}\right)}$$

Thus

$$d\left(e_\ell\, d^2\underset{\sim}{\Sigma}^\ell\right) = e_n\Gamma^n{}_{\ell r}\,g^{\ell k}\,dx^r\wedge\sqrt{g}\,[kij]\frac{dx^i\wedge dx^j}{2!} + ② + ③$$

$$\delta^r{}_k\sqrt{g}\;dx^1\wedge dx^2\wedge dx^3$$

$$= \underbrace{e_n\Gamma^n{}_{\ell r}\,g^{\ell r}\sqrt{g}\;dx^1\wedge dx^2\wedge dx^3}_{①} + ② + ③ = ① + ② + ③$$

$$① = e_n\,\tfrac{1}{2}g^{nm}\left(g_{m\ell,r} + g_{mr,\ell} - g_{r\ell,m}\right)g^{\ell r}\sqrt{g}\;dx^1\wedge dx^2\wedge dx^3$$

$$= e_n\left(g^{nm}g_{m\ell,r}\,g^{\ell r} - \tfrac{1}{2}g^{nm}g^{\ell r}g_{r\ell,m}\right)\sqrt{g}\;dx^1\wedge dx^2\wedge dx^3$$

$$② = e_\ell(-)g^{\ell r}g^{sk}dg_{rs}\;\sqrt{g}\,[kij]\;dx^i\wedge dx^j/2!$$

$$= -e_\ell\,g^{\ell r}g^{sk}g_{rs,p}\sqrt{g}\,[kij]\,dx^p\wedge dx^i\wedge dx^j/2!$$

$$= -e_\ell\,g^{\ell r}g^{sk}g_{rs,p}\sqrt{g}\;\delta^p{}_k\;dx^1\wedge dx^2\wedge dx^3$$

$$= -e_\ell\,g^{\ell r}g^{sp}g_{rs,p}\sqrt{g}\;dx^1\wedge dx^2\wedge dx^3$$

$$= -e_n\,g^{nm}g^{sp}g_{ms,p}\sqrt{g}\;dx^1\wedge dx^2\wedge dx^3$$

$$
\begin{aligned}
③ &= e_\ell\, g^{\ell k} \frac{1}{\sqrt{g}}\, d\sqrt{g} \wedge [k\,i\,j]\, dx^i \wedge dx^j / 2! \\
&= e_\ell\, g^{\ell k} \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^p} [k\,i\,j]\, dx^p \wedge dx^i \wedge dx^j / 2! \\
&= e_\ell\, g^{\ell k} \frac{1}{2}\, g^{ms}\, g_{ms,p}\, \delta^p_k \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3 \\
&= e_\ell\, g^{\ell k} \frac{1}{2}\, g^{ms}\, g_{ms,k} \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3 \;, \\
&= e_n\, g^{nk} \frac{1}{2}\, g^{ms}\, g_{ms,k} \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3
\end{aligned}
$$

$$
\begin{aligned}
①+②+③ = {} & e_n \left( g^{nm}\, g_{m\ell,r}\, g^{\ell r} - \frac{1}{2} g^{nm}\, g^{\ell r}\, g_{r\ell,m} \right) \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3 \\
& - e_n\, g^{nm}\, g^{sp}\, g_{ms,p} \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3 \\
& + e_n\, g^{nk} \frac{1}{2}\, g^{\ell r}\, g_{\ell r,k} \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3 \\
= {} & 0
\end{aligned}
$$

Thus, in terms of the original notation one has

$$
d\left(e_\ell\, d^2 \underset{\approx}{\Sigma}^\ell\right) = 0
$$

$$
d\left({}^3\vec{\Sigma}\right) = 0
$$

$$
d\left(e_\ell\, \epsilon^\ell{}_{ij}\, dx^i \wedge dx^j\right) = 0
$$