Lecture 11

Particle conservation
mathematized

I. Particle world lines in a particle world tube

II. Differential law of particle conservation

III. Coordinate invariant volume

In MTW grasp
the ideas in

Box 4.4
Box 5.1
Fig 5.1
Box 5.2

(11.1)

I. Particle World Tube

The spacetime history of particles is geometrized by their world lines. Particles which do not go out of existence have world lines that do not terminate.
An aggregate of non-colliding particles having the same observed 4-velocity in a local spacetime region form a world tube which is filled with the world lines of these particles.

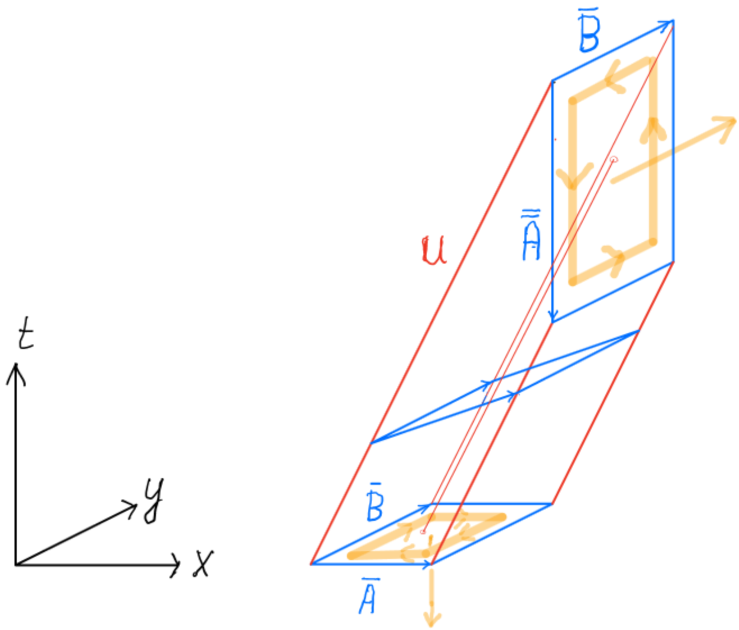


Figure 11.1 World tube composed type u particle world lines with 3-d cross sections $(\bar{A}, \bar{B}, \bar{C})$ and $(\bar{\bar{A}}, \bar{B}, \bar{C})$ which respectively are space like and time like elements of 3-volume. These volume elements contain the same number of particles. This is because the particle world lines do not terminate as they evolve from $(\bar{A}, \bar{B}, \bar{C})$ to $(\bar{\bar{A}}, \bar{B}, \bar{C})$. This is a geometrical statement of particles not being destroyed (or created).

The number observed particles is geometrized by the number of worldlines that cut through sections across the world tube. As depicted in Figure 11.1, a cross section is spanned by three 4-vectors such as $\bar{A}$, $\bar{B}$, and $\bar{C}$ or $\bar{\bar{A}}$, $\bar{B}$, and $\bar{C}$. A typical cross section such as $(\bar{A}, \bar{B}, \bar{C})$ or (A, B, C) consists of three space like vectors. They span a spatial element of

(11.3)

volume which contains the number of particles

$$^*\mathbf{S}(\bar{A},\bar{B},\bar{C}) = \# = {}^*\mathbf{S}(A,B,C) \qquad (11.1a)$$

$$N u^{\bar\mu}\, \epsilon_{\bar\mu\bar\alpha\bar\beta\bar\gamma}\, d\bar{x}^{\bar\alpha}\wedge d\bar{x}^{\bar\beta}\wedge d\bar{x}^{\bar\gamma}(\bar{A},\bar{B},\bar{C}) = \# = N u^{\mu}\, \epsilon_{\mu\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}(A,B,C). \qquad (11.1b)$$

On the other hand, a time like cross section $(\bar{\bar{A}},\bar{B},\bar{C})$ where one of its spanning vectors, $\bar{\bar{A}}$, is time like, contains the number of particles

$$^*\mathbf{S}(\bar{\bar{A}},\bar{B},\bar{C}) = \# = N u^{\bar\mu}\, \epsilon_{\bar\mu\bar\alpha\bar\beta\bar\gamma}\, d\bar{x}^{\bar\alpha}\wedge d\bar{x}^{\bar\beta}\wedge d\bar{x}^{\bar\gamma}(\bar{\bar{A}},\bar{B},\bar{C}). \qquad (10.2)$$

They flow across the spatial opening of area $\vec{\bar{B}}\times\vec{\bar{C}}$ during the time interval $\Delta\bar{t}$ of the time like 4-vector

$$\bar{\bar{A}} = -\Delta\bar{t}\,\frac{\partial}{\partial\bar{x}^{\bar 0}} + 0\,\frac{\partial}{\partial\bar{x}^{\bar 1}} + 0\,\frac{\partial}{\partial\bar{x}^{\bar 2}} + 0\,\frac{\partial}{\partial\bar{x}^{\bar 3}}$$

There is no creation nor destruction of the type *u* particles that make up the type *u* world tube depicted in Figure 10.1.

Consequently, the particle # in Eqs.(11.1) and (11.2) are the same:

(11,4)

$$\boxed{N u^{\bar\mu}\,\epsilon_{\bar\mu\bar\alpha\bar\beta\bar\gamma}\,d\bar x^{\bar\alpha}\wedge d\bar x^{\bar\beta}\wedge d\bar x^{\bar\gamma}(\bar A,\bar B,\bar C) = \# = N u^{\bar\mu}\,\epsilon_{\bar\mu\bar\alpha\bar\beta\bar\gamma}\,d\bar x^{\bar\alpha}\wedge d\bar x^{\bar\beta}\wedge d\bar x^{\bar\gamma}(\bar{\bar A},\bar B,\bar C)}$$

Equation (11.3) is a statement of the law of particle conservation in the 4-d domain inside the world tube depicted by Figure 11.1. (11.3)

II. Differential Law of Particle Conservation

The local law of particle conservation depicted in Figure 11.1 also holds globally as depicted in Figure 11.2. There the type *u* particle world tube connecting the initial with the final 3-volumes consists of world lines of whatever Nature dictates.

The global mathematization of particle conservation, as in Figure 11.2, is accomplished by applying the 3-4 version of Green's Theorem to the (local) differential law of particle conservation.

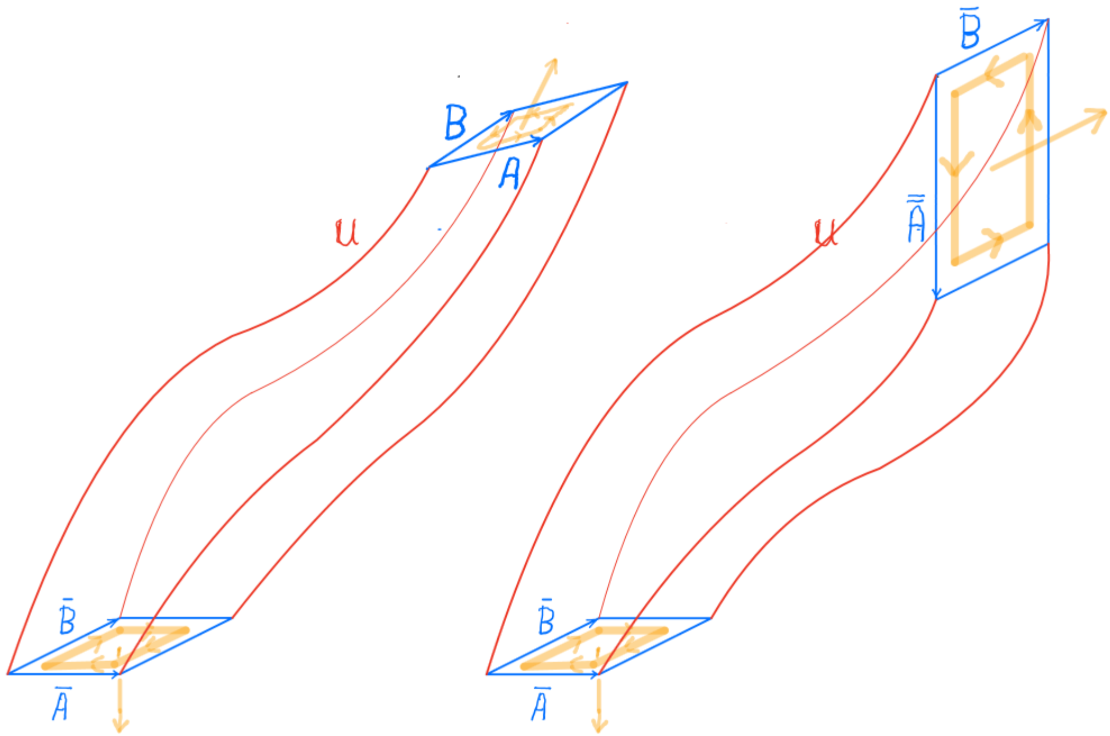


Figure 11.2 World tubes with the common purely space like initial cross section $(\bar{A}, \bar{B}, \bar{C})$, but with different final cross sections; namely, (i) (A, B, C), which is pure space like as depicted in panel (a), and (ii) $(\bar{\bar{A}}, \bar{B}, \bar{C})$, which is time like because $\bar{\bar{A}}$, a time like vector depicted in panel (b), is one of the three that span the volume.

The line of reasoning leading to the local differential law of particle conservation is four-step process. (11.5)

Step 1. ("The volume vector")

Introduce in 4-D spacetime what in 3-D Euclidean space is the "bivector" or cross-product:

a) In 3-D space

$$(\vec{A}\times\vec{B})_j = \epsilon_{jk\ell} A^k B^\ell \qquad \text{(covector components)}$$

$$\begin{aligned}\vec{A}\times\vec{B} &= \vec{e}_i\, g^{ij}\epsilon_{jk\ell} A^k B^\ell \\ &= \vec{e}_i\, \epsilon^i{}_{k\ell} A^k B^\ell \\ &= \vec{e}_i\, \epsilon^i{}_{k\ell}\, dx^k \wedge dx^\ell/2!\,(\vec{A}\,\vec{B}) \qquad (11.4)\end{aligned}$$

The vectorial differential 2-form

$$\boxed{{}^{(2)}\Sigma \equiv \vec{e}_i\, {}^{(2)}\underset{\approx}{\Sigma}{}^i \equiv \vec{e}_i\, \epsilon^i{}_{k\ell}\, dx^k\wedge dx^\ell/2!} \qquad (11.5)$$

is a tensor field of rank $\binom{1}{2}$

It has the property that it is constant under parallel transport into any direction:

$$\boxed{d({}^{(2)}\Sigma) \equiv d(e_i\, g^{ij}\epsilon_{jk\ell}\, dx^k\wedge dx^\ell) = 0} \qquad (11.6)$$

b) In 4-D space *

$$\star(A\wedge B\wedge C)_\nu = \epsilon_{\nu\alpha\beta\gamma} A^\alpha B^\beta C^\gamma \qquad (11.7)$$

$$\star(A\wedge B\wedge C) = e_\nu\, g^{\nu\sigma}\epsilon_{\sigma\alpha\beta\gamma}A^\alpha B^\beta C^\gamma \qquad (11.6)$$

$$= e_\nu\, \epsilon^\nu{}_{\alpha\beta\gamma}A^\alpha B^\beta C^\gamma \qquad (11.8)$$

$$= e_\nu\, \epsilon^\nu{}_{\alpha\beta\gamma}\, dx^\alpha\wedge dx^\beta\wedge dx^\gamma/3!\,(A,B,C)$$

Definition ("Volume vector")

The vectorial 3-form

$$\boxed{{}^{(3)}\Sigma \equiv e_\nu\, {}^{(3)}\underset{\sim}{\Sigma}{}^\nu \equiv e_\nu\, \epsilon^\nu{}_{\alpha\beta\gamma}\, dx^\alpha\wedge dx^\beta\wedge dx^\gamma/3!} \qquad (11.9)$$

is called the <u>3-volume vector in 4 dimensions.</u>

- It is a tensor field of rank $\binom{1}{3}$.
- It is the vector perpendicular to the volume spanned by three as-yet-unspecified 4-D vectors.
- Its magnitude is a measure the spanned volume.
- It has the property that it is constant under parallel transport into any direction in 4-D:

$$\boxed{d\left({}^{(3)}\Sigma\right) = d\left(e_\nu\, g^{\nu\sigma}\epsilon_{\sigma\alpha\beta\gamma}\, dx^\alpha\wedge dx^\beta\wedge dx^\gamma/3!\right) = 0}$$

(11.10)

11.7

*\ footnote {The mapping

$$\star : V\wedge V\wedge V \xrightarrow{\star} V,$$

defined by Eq. (11.8), is the same as MTW's Eq. (15.15), except that their's is

$$\star(A\wedge B\wedge C) = A^{\alpha}B^{\beta}C^{\gamma}\,\epsilon_{\alpha\beta\gamma}{}^{\nu}\,e_{\nu},$$

which differs from ours only by a change in sign.

Step 2. ("The matter-volume decomposition")

Recalling the particle 4-current

$$\mathbf{S} = N\mathbf{u} = \underbrace{Nu^{\mu}}_{S^{\mu}}e_{\mu},$$

reformulate the scalar density-flux 3-form, Eq. (11.1) as the inner product 3-form

$$^{*}\mathbf{S} = \mathbf{S}\cdot{}^{(3)}\Sigma.$$

11.8

Indeed, a notational computation based on the boxed definition, Eq. (11.9) on page 11.5, yields

$$
\begin{aligned}
{}^*\mathbf{S} &= N u^{\mu} \; \epsilon_{\mu\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}/3! \\
&= S^{\mu} \; g_{\mu\nu}\, \underbrace{\epsilon^{\nu}{}_{\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}/3!}_{\equiv\ {}^{(3)}\underset{\sim}{\Sigma}^{\nu}} \\
&= S^{\mu} \mathbf{e}_{\mu}\cdot\mathbf{e}_{\nu}\, {}^{(3)}\underset{\sim}{\Sigma}^{\nu} \\
&= \mathbf{S}\cdot{}^{(3)}\Sigma .
\end{aligned}
$$

$$
\boxed{{}^*\mathbf{S} = \left(\mathbf{e}_{\sigma} S^{\sigma}\right)\cdot\left(\mathbf{e}_{\mu}\, {}^{(3)}\underset{\sim}{\Sigma}^{\mu}\right)} \qquad (11.11)
$$

This inner product decomposition of the particle density-flux 3-form mathematizes the conceptual separation between (i) the nature of matter (here its four-current **S**) and (ii) the geometrical space (here its 3-volume ${}^{(3)}\Sigma$) available for its occupation.

Step 3. ("The differential law") 11.9

Take exterior derivative d of Eq. (11.11) and find

$$d\,{}^{*}\mathcal{S} = d\left[S^{\mu} \epsilon_{\mu\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}\right]$$
$$= d\left[S^{\mu} \delta^{\nu}_{\mu} \epsilon_{\nu\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}\right]$$
$$= d\left[S^{\mu} g_{\mu\nu} g^{\nu\sigma} \epsilon_{\sigma\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}\right]$$
$$= d\left[S^{\mu} e_{\mu}\cdot e_{\nu} \epsilon^{\nu}{}_{\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}\right]$$
$$= d\left[\mathcal{S}\cdot {}^{(3)}\Sigma \right] \qquad (11.12)$$

The exterior derivative of this product* is

$$d\,{}^{*}\mathcal{S} = d(S^{\sigma} e_{\sigma}) \wedge\cdot (e_{\nu} {}^{(3)}\underset{\sim}{\Sigma}{}^{\nu}) + S^{\mu} e_{\mu}\cdot d(e_{\nu} {}^{(3)}\underset{\sim}{\Sigma}{}^{\nu}) \qquad (11.13)$$

\footnote{
Exterior product and interior product are commutative operations, i.e. "$\wedge\cdot$" = "$\cdot\wedge$". This is because in exterior algebra the coefficients may also be those of a vector (or tensor) field besides those of a mere scalar field. Thus "$\wedge\cdot\vec{v}$" = "$\cdot\vec{v}\wedge$". The operation "$\wedge$" and "$\cdot$" are freely interchangeable.}

(11.10)

The second term vanishes because vectorial 3-volume form is constant, $d({}^{(3)}\Sigma) \equiv d(e_\nu {}^{(3)}\Sigma^\nu) = 0$.
To evaluate the first term, start with (i), the fact that the differential of the vector $S = e_\sigma S^\sigma$ is

$$\begin{aligned} d(e_\sigma S^\sigma) &= e_\sigma\, dS^\sigma + S^\sigma de_\sigma \\ &= e_\sigma \frac{\partial S^\sigma}{\partial x^\mu} dx^\mu + e_\nu S^\sigma \Gamma^\nu_{\ \sigma\mu} dx^\mu \\ &= e_\sigma \left(S^\sigma_{\ ,\mu} + S^\nu \Gamma^\sigma_{\ \nu\mu} \right) dx^\mu \\ &\equiv e_\sigma S^\sigma_{\ ;\mu}\, dx^\mu . \end{aligned} \qquad (11.14)$$

Here $S^\sigma_{\ ;\mu}$ are the component of the covariant derivative of the particle 4-current S.

Then (ii), take advantage of the fact that the wedge product of $dx^\mu \wedge e_\nu {}^{(3)}\Sigma^\nu$ simplifies considerarably:

$$\begin{aligned} dx^\mu \wedge e_\nu {}^{(3)}\Sigma^\nu &= e_\nu\ dx^\mu \wedge \epsilon^\nu_{\ \alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3! \\ &= e_\nu\ g^{\nu\rho} \epsilon_{\rho\alpha\beta\gamma}\ dx^\mu \wedge dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3! \\ &= e_\nu\ g^{\nu\rho} \delta^\mu_\rho \sqrt{-g}\ dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{aligned} \qquad (11.15)$$

Applying Eqs. (1.14)-(11.15) to (11.13)

$$\begin{aligned} d({}^*S) &= e_\sigma S^\sigma_{\ ;\mu} \cdot e_\nu\, g^{\nu\rho} \delta^\mu_\rho \sqrt{-g}\ dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ &= S^\sigma_{\ ;\sigma} \sqrt{-g}\ dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{aligned}$$

11.11

Both the covariant divergence

$$S^\sigma{}_{;\sigma} = \frac{1}{\sqrt{-g}} \frac{\partial (S^\sigma \sqrt{-g})}{\partial x^\sigma},$$

and the 4-volume element $\sqrt{-g}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ are coordinate frame invariants.

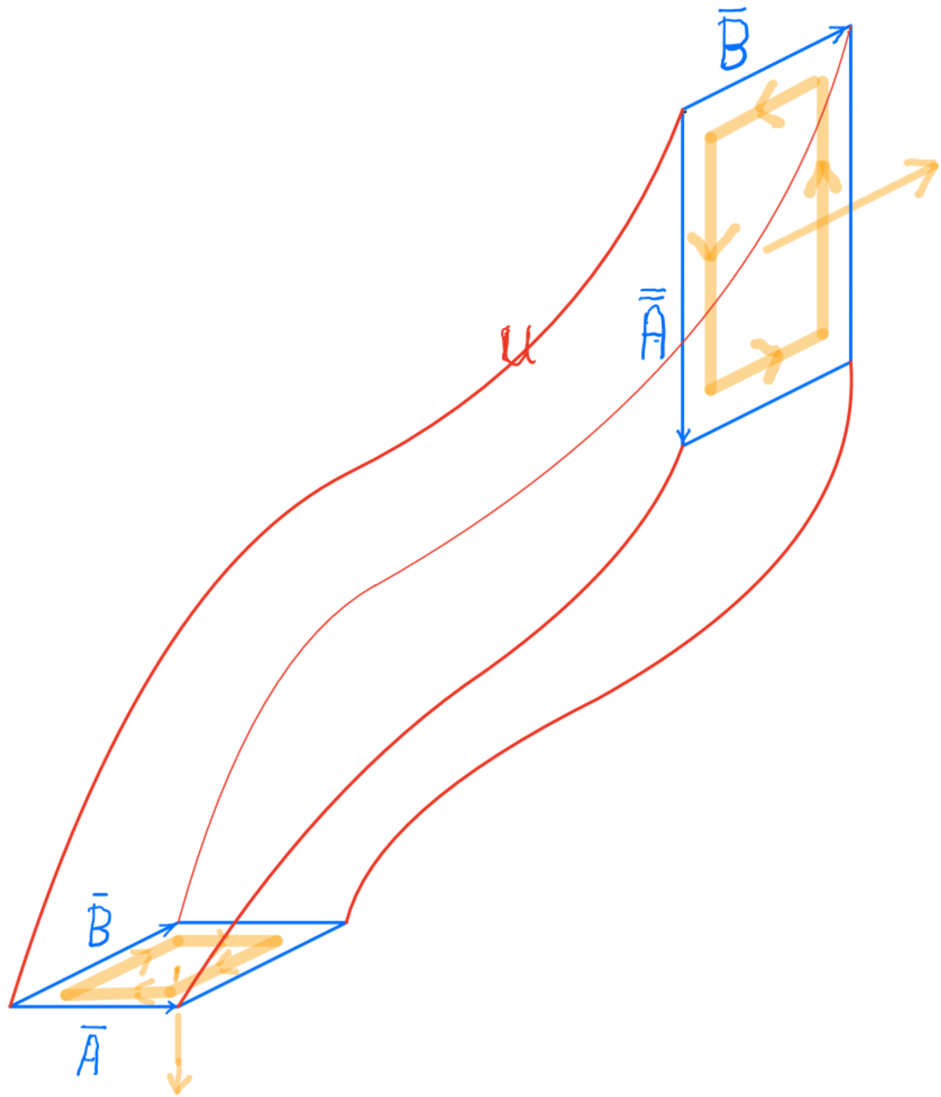
Consequenly,

$$d({}^*S) = \frac{\partial (S^\sigma \sqrt{-g})}{\partial x^\sigma}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \qquad (11.16)$$

$$= \left(\begin{array}{l} \#\text{ of particles created in} \\ \text{the invariant spacetime} \\ \text{4-volume } \sqrt{-g}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{array} \right)$$

Whenever particles are neither created nor destroyed, then such a state of affairs is mathematized by the statement

$$\boxed{\frac{\partial (S^\sigma \sqrt{-g})}{\partial x^\sigma} = 0} \quad \left(\begin{array}{l} \text{"Conservation} \\ \text{of particles"} \end{array} \right) \qquad (11.17)$$


$\bar{B}$
$\bar{\bar{A}}$
$\mathcal{U}$
$\bar{B}$
$\bar{A}$

III. Coordinate invariant volume

(11.12)

An element of volume is rooted in two key concepts: that of the Jacobian and that of the Levi-Civita Tensor.

1.) The Jacobian

Consider the 4-volume element which is spanned by the tetrad of 4-vectors

$$
\begin{aligned}
\bar{A} &= \Delta\bar{x}^{\bar 0}\frac{\partial}{\partial\bar{x}^{\bar 0}} + 0\,\frac{\partial}{\partial\bar{x}^{\bar 1}} + 0\,\frac{\partial}{\partial\bar{x}^{\bar 2}} + 0\,\frac{\partial}{\partial\bar{x}^{\bar 3}} = \bar{A}^{\bar\alpha}\frac{\partial}{\partial\bar{x}^{\bar\alpha}}\\
\bar{B} &= 0\,\frac{\partial}{\partial\bar{x}^{\bar 0}} + \Delta\bar{x}^{\bar 1}\frac{\partial}{\partial\bar{x}^{\bar 1}} + 0\,\frac{\partial}{\partial\bar{x}^{\bar 2}} + 0\,\frac{\partial}{\partial\bar{x}^{\bar 3}} = \bar{B}^{\bar\beta}\frac{\partial}{\partial\bar{x}^{\bar\beta}}\\
\bar{C} &= 0\,\frac{\partial}{\partial\bar{x}^{\bar 0}} + 0\,\frac{\partial}{\partial\bar{x}^{\bar 1}} + \Delta\bar{x}^{\bar 2}\frac{\partial}{\partial\bar{x}^{\bar 2}} + 0\,\frac{\partial}{\partial\bar{x}^{\bar 3}} = \bar{C}^{\bar\gamma}\frac{\partial}{\partial\bar{x}^{\bar\gamma}}\\
\bar{D} &= 0\,\frac{\partial}{\partial\bar{x}^{\bar 0}} + 0\,\frac{\partial}{\partial\bar{x}^{\bar 1}} + 0\,\frac{\partial}{\partial\bar{x}^{\bar 2}} + \Delta x^{3}\frac{\partial}{\partial\bar{x}^{\bar 3}} = \bar{D}^{\delta}\frac{\partial}{\partial\bar{x}^{\bar\delta}}
\end{aligned}
\qquad (11.18)
$$

namely

$$
\Delta\bar{x}^{0}\Delta\bar{x}^{\bar 1}\Delta\bar{x}^{\bar 2}\Delta\bar{x}^{\bar 3} = [\bar\alpha\,\bar\beta\,\bar\gamma\,\bar\delta]\;\bar{A}^{\bar\alpha}\bar{B}^{\bar\beta}\bar{C}^{\bar\gamma}\bar{D}^{\bar\delta} = \det\begin{vmatrix}\Delta\bar{x}^{\bar 0} & 0 & 0 & 0\\ 0 & \Delta\bar{x}^{\bar 1} & 0 & 0\\ 0 & 0 & \Delta\bar{x}^{\bar 2} & 0\\ 0 & 0 & 0 & \Delta\bar{x}^{\bar 3}\end{vmatrix}
$$

Here

$$
[\bar\alpha\,\bar\beta\,\bar\gamma\,\bar\delta] = \begin{cases} +1 \text{ when } \bar\alpha\bar\beta\bar\gamma\bar\delta \text{ is an even permutation of } 0123\\ -1 \text{ when } \bar\alpha\bar\beta\bar\gamma\bar\delta \text{ is an odd permutation of } 0123\\ 0 \text{ when any two indeces repeat}\end{cases}
$$

are the components of the Levi-Civita tensor relative to the rectilinear coordinate system $\{\bar{x}^{\bar\mu}\}$.

Relative to the curvilinear coordinate system $\{y^{\mu}\}$ this element of 4-volume is

$$\Delta\bar{x}^{\bar{0}}\Delta\bar{x}^{\bar{1}}\Delta\bar{x}^{\bar{2}}\Delta\bar{x}^{\bar{3}} = [\bar{\alpha}\,\bar{\beta}\,\bar{\gamma}\,\bar{\delta}]\frac{\partial\bar{x}^{\bar{\alpha}}}{\partial y^{\mu}}\frac{\partial\bar{x}^{\bar{\beta}}}{\partial y^{\nu}}\frac{\partial\bar{x}^{\bar{\gamma}}}{\partial y^{\rho}}\frac{\partial\bar{x}^{\bar{\delta}}}{\partial y^{\sigma}}\frac{\partial y^{\mu}}{\partial\bar{x}^{\bar{0}}}\frac{\partial y^{\nu}}{\partial\bar{x}^{\bar{1}}}\frac{\partial y^{\rho}}{\partial\bar{x}^{\bar{2}}}\frac{\partial y^{\sigma}}{\partial\bar{x}^{\bar{3}}}\Delta\bar{x}^{\bar{0}}\Delta\bar{x}^{\bar{1}}\Delta\bar{x}^{\bar{2}}\Delta\bar{x}^{\bar{3}}$$

$$= \det\left[\frac{\partial\bar{x}^{\bar{\alpha}}}{\partial y^{\mu}}\right][\mu\,\nu\,\rho\,\sigma]\;\frac{\partial y^{\mu}}{\partial\bar{x}^{\bar{0}}}\frac{\partial y^{\nu}}{\partial\bar{x}^{\bar{1}}}\frac{\partial y^{\rho}}{\partial\bar{x}^{\bar{2}}}\frac{\partial y^{\sigma}}{\partial\bar{x}^{\bar{3}}}\Delta\bar{x}^{\bar{0}}\Delta\bar{x}^{\bar{1}}\Delta\bar{x}^{\bar{2}}\Delta\bar{x}^{\bar{3}} \qquad (11.13)$$

$$\equiv \epsilon_{\mu\nu\rho\sigma}\, dy^{\mu}\wedge dy^{\nu}\wedge dy^{\rho}\wedge dy^{\sigma}/4!\;\left(\frac{\partial}{\partial\bar{x}^{\bar{0}}},\frac{\partial}{\partial\bar{x}^{\bar{1}}},\frac{\partial}{\partial\bar{x}^{\bar{2}}},\frac{\partial}{\partial\bar{x}^{\bar{3}}}\right)\Delta\bar{x}^{\bar{0}}\Delta\bar{x}^{\bar{1}}\Delta\bar{x}^{\bar{2}}\Delta\bar{x}^{\bar{3}}$$

The determinant, $\det\left[\frac{\partial\bar{x}^{\bar{\alpha}}}{\partial y^{\mu}}\right]$, of the matrix of partial derivatives $\frac{\partial\bar{x}^{\bar{\alpha}}}{\partial y^{\mu}}$ is the Jacobian of the coordinate transformation $\{y^{\mu}\}\to\{\bar{x}^{\bar{\alpha}}\}$.

2.) The Levi-Civita Tensor.

Furthermore,

$$\epsilon_{\mu\nu\rho\sigma} = \det\left[\frac{\partial\bar{x}^{\bar{\alpha}}}{\partial y^{\mu}}\right][\mu\,\nu\,\rho\,\sigma] \qquad (11.19)$$

are the components of the Levi-Civita tensor

$$\epsilon = \epsilon_{\mu\nu\rho\sigma}\, dy^{\mu}\wedge dy^{\nu}\wedge dy^{\rho}\wedge dy^{\sigma}/4!\,, \qquad (11.20)$$

The defining property of ϵ is that it is the multilinear map which assigns a numerical measure to the 4-volume spanned by an arbitrarily tetrad of vectors A, B, C, D. Its additional key property is that it assigns an orientation to such a tetrad of vectors. Thus, the ordered tetrads (A,B,C,D) and (B,A,C,D) are distinguishable by their orientation because

$$\epsilon(A,B,C,D) = -\epsilon(B,A,C,D).$$

It is understood, but nevertheless worth pointing out, that the mathematization of the size and orientation of the volume element spanned by a tetrad of 4-D vectors, such as those in Eq. (11.18), does not depend on one's knowledge of their metric properties, even if they have any. (11.14)

However, if a metric tensor is given, both the Jacobian determinant and the Levi-Civita tensor are determined. For a given metric tensor

$$g = \eta_{\bar\alpha\bar\beta}\, d\bar x^{\bar\alpha} \otimes d\bar x^{\bar\beta} = g_{\mu\nu}\, dy^\mu \otimes dy^\nu \qquad (11.21)$$

the determinant of its components are readily related to the Jacobian determinant $\det\left[\frac{\partial \bar x^{\bar\alpha}}{\partial y^\mu}\right]$. Indeed, Eq. (11.21) implies

$$g_{\mu\nu} = \frac{\partial \bar x^{\bar\alpha}}{\partial y^\mu}\, \eta_{\bar\alpha\bar\beta}\, \frac{\partial \bar x^{\bar\beta}}{\partial y^\nu}$$

or in matrix notation

$$[g_{\mu\nu}] = \left[\frac{\partial \bar x^{\bar\alpha}}{\partial y^\mu}\right]^{tr} [\eta_{\bar\alpha\bar\beta}] \left[\frac{\partial \bar x^{\bar\beta}}{\partial y^\nu}\right], \qquad [\eta_{\bar\alpha\bar\beta}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It follows that

$$\det[g_{\mu\nu}] \equiv g = \det\left[\frac{\partial \bar x^{\bar\alpha}}{\partial y^\mu}\right](-1)\det\left[\frac{\partial \bar x^{\bar\alpha}}{\partial y^\mu}\right]$$

or

$$\boxed{\det\left[\frac{\partial \bar x^{\bar\alpha}}{\partial y^\mu}\right] = \sqrt{-g}}$$

(11.15)

Apply this metric-based Jacobian determinant to the Levi-Civita tensor, Eqs. (11.19)-(11.20) and obtain

$$\boxed{\epsilon = \sqrt{-g}\,[\mu\nu\rho\sigma]\; dy^\mu \wedge dy^\nu \wedge dy^\rho \wedge dy^\sigma / 4!}$$

where

$$\boxed{\sqrt{-g}\,[\mu\nu\rho\sigma] = \epsilon_{\mu\nu\rho\sigma}}$$

are the components of ϵ relative to the curvilinear coordinate basis $\{dy^\mu\}$.