Lecture 12

The Law of Momenergy Conservation

I. Momenergy For A Mixture Of Particles

II. The momergy density-flux 3-form $^*\mathbb{T}$.

III. Conservation of momenergy via $^*\mathbb{T} \Longrightarrow d{}^*\mathbb{T}=0$

In MTW read (i.e. grasp) § 5.4 and Box 5.4

In Wheeler's A JOURNEY INTO GRAVITY AND SPACETIME read Chapter 6 on Momenergy. This very readable chapter particularizes and conceptualizes the phyical basis of momenergy and its conservation.

12.1

I. Momenergy of a Mixture of Particles

The 19th century atomic theory of matter, as well as its 20th century version, condense the fact that, within the relevant context, matter comes in the form of different kinds of particles, and particles have the attribute of being carriers of momenergy. Hence the question: "What is the momenergy carried by an aggregate of different particles moving with different 4-velocities?"

Answer: Consider 3-D spacetime element of volume spanned by a triad of 4-D vectors (A, B, C) and populated by different particles having their own respective

rest masses $m_1, m_2, \ldots$

4-velocities $u_1, u_2, \ldots$

momenergies $p_1, p_2, \ldots$

invariant ("proper") densities $N_1, N_2, \ldots$

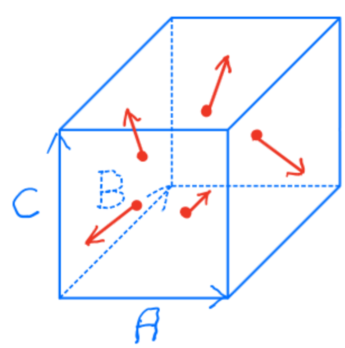

Figure 12.1 Particles with their respective 4-velocities in a 3-D volume element spanned by the triad (A,B,C) of 4-D vectors one of which may be time-like.

The total amount of momenergy of these particles in this (A,B,C)-spanned volume element is the sum of the contributions, $p_a {}^*S_a(A,B,C)$, from each particle species.

$$
\begin{aligned}
{}^*T(A,B,C) &= \sum_a p_a\, {}^*S_a(A,B,C) \\
&= \sum_a p_a N_a u_a^\nu\, \epsilon_{\nu\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3!\,(A,B,C) \\
&= e_\mu \underbrace{\sum_a p_a^\mu N_a u_a^\nu}_{T^{\mu\nu}} \overbrace{e_\nu \cdot e_\sigma}^{g_{\nu\sigma}} \underbrace{\epsilon^\sigma{}_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3!\,(A,B,C)}_{e_\sigma\, {}^{(3)}\underset{\sim}{\Sigma}{}^\sigma \equiv {}^{(3)}\Sigma^\sigma}
\end{aligned}
$$

II. The Momenergy Density-Flux 3-form ${}^*T$

Following the introduction of the 3-volume

$$
{}^{(3)}\Sigma = e_\sigma\, {}^{(3)}\underset{\sim}{\Sigma}{}^\sigma = e_\sigma\, \epsilon^\sigma{}_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3!
$$

in Eq.(11.11), page 11.8 of Lecture 11, find that here, just as there, the density-flux has the same inner product decomposition

$$
{}^*T(A,B,C) = e_\mu T^{\mu\nu} e_\nu \cdot e_\sigma\, {}^{(3)}\underset{\sim}{\Sigma}{}^\sigma (A,B,C)
$$

This holds for all triads of 4-vectors (A,B,C). Consequenly, the <u>momenergy density-flux</u> is mathematized by

$$
\boxed{{}^*\mathbb{T} = \mathbb{T}\cdot {}^{(3)}\Sigma} \qquad (12.1)
$$

Here $^{*}\mathbb{T} = \mathbf{e}_\mu \sum_a p_a^\mu N_a u_a^\nu \, \epsilon_{\nu\alpha\beta\gamma} \, dx^\alpha \wedge dx^\beta \wedge dx^\gamma / 3!$, (12.2) 12,3

a tensor of rank $\binom{1}{3}$,

is the momenrgy density-flux, where

$$T^{\mu\nu} = \sum_a p_a^\mu N_a u_a^\nu$$

are the components of the momenergy 4-current

$$\boxed{\mathbb{T} = \mathbf{e}_\mu \otimes T^{\mu\nu} \mathbf{e}_\nu}, \qquad (12.3)$$

a tensor of rank $\binom{2}{0}$, and

$${}^{(3)}\underset{\sim}{\Sigma}{}^{\sigma} = \epsilon^{\sigma}{}_{\alpha\beta\gamma} \, dx^\alpha \wedge dx^\beta \wedge dx^\gamma$$

are the components of the invariant vectorial 3-volume measure

$$\boxed{{}^{(3)}\Sigma = \mathbf{e}_\sigma \, {}^{(3)}\underset{\sim}{\Sigma}{}^{\sigma}}, \qquad (12.4)$$

which is a tensor of rank $\binom{1}{3}$.

The inner product decomposition Eq.(12.1) of the invariant momenergy density-flux ($^{*}\mathbb{T}$, Eq.(12.2)) integrates mathematically and invariantly two disjoint concepts,

(i) the momenergy property ($\mathbb{T}$, Eq.(12.3)) of matter, and

(ii) the 3-volume measure (${}^{(3)}\Sigma$, Eq.(12.4)) of any 3-d box bounded

12.4

by a triad (A, B, C) of 4-d vectors.

The 3-volume measure, Eq. (12.4), is compatible with metric-induced parallel transport. This means ${}^{(3)}\Sigma$ does not change under parallel trans- into any direction. This fact is mathematized by the vanishing of its exterior derivative

$$d\left({}^{(3)}\Sigma\right) = 0 \qquad (12.5)$$

in the same way as the compatibility between the metric $g = g_{\mu\nu}\, dx^\mu \otimes dx^\mu$ and parallel transport, namely

$$d(g) = 0.$$

III. Conservation of Momenergy

(12.5)

The inner product decomposition of the density-flux of matter is the same for particle number (or charge)

$$^*\mathbf{S} = \mathbf{S}\cdot{}^{(3)}\Sigma$$

and for momenergy

$$^*\mathbb{T} = \mathbb{T}\cdot{}^{(3)}\Sigma.$$

Consequently, the evaluation of $d(^*\mathbb{T})$ parallels that of $d(^*S)$ in Lecture 11:

$$
\begin{aligned}
d(^*\mathbb{T}) &= d(\mathbb{T}\cdot{}^{(3)}\Sigma) \\
&= \quad d(\mathbb{T}) \quad \wedge\cdot{}^{(3)}\Sigma + \mathbb{T}\cdot\wedge d({}^{(3)}\Sigma) \\
&= d(T^{\tau\sigma}\mathbf{e}_\tau\otimes\mathbf{e}_\sigma)\wedge\cdot{}^{(3)}\Sigma + \quad 0 \qquad (12.6)
\end{aligned}
$$

The covariant differential of $\mathbb{T}$ follows the same rules* as those of $\mathbf{S}$, Eq. (11.14) in Lecture 11. Thus,

$$d(T^{\tau\sigma}\mathbf{e}_\tau\otimes\mathbf{e}_\sigma) = e_\tau\otimes e_\sigma\, T^{\tau\sigma}{}_{;\mu}\,dx^\mu. \qquad (12.7)$$

Here $T^{\tau\sigma}{}_{;\mu}$ are the components of the covariant derivative of the momenergy 4-current $\mathbb{T}$, a.k.a. the energy-momentum tensor. As in Eq. (11.15), the wedge product $dx^\mu\wedge e_\nu{}^{(3)}\underset{\approx}{\Sigma}$ simplifies considerably:

$$
\begin{aligned}
dx^\mu\wedge e_\nu{}^{(3)}\Sigma^\nu &= dx^\mu\wedge e_\nu\,\epsilon^\nu{}_{\alpha\beta\gamma}\,dx^\alpha\wedge dx^\beta\wedge dx^\gamma/3! \\
&= e_\nu\, g^{\nu\rho}\,\epsilon_{\rho\alpha\beta\gamma}\,dx^\mu\wedge dx^\alpha\wedge dx^\beta\wedge dx^\gamma/3! \\
&= e_\nu\, g^{\nu\rho}\sqrt{-g}\,\delta^\mu_\rho\, dx^0\wedge dx^1\wedge dx^2\wedge dx^3 \qquad (12.8)
\end{aligned}
$$

\footnote{ There are two types of products in (12.6) working with tensor-valued ($^*\mathbb{T}$), vector-valued ($^{(3)}\Sigma$), and scalar-valued ($^*\mathbb{S}$) three-forms, namely the wedge product "$\wedge$" between differential forms and (ii) the inner product "$\cdot$" between a vector and a vector, or a tensor (i.e. a "bivector") and a vector, or the simple product of a scalar with a vector / tensor.

All these objects (scalars, vectors, tensors) are to be viewed as mere coefficients in the space of linear 3-forms, whose basis is $\{dx^\alpha \wedge dx^\beta \wedge dx^\gamma\}$. These scalar, vector and tensor-valued 3-forms are multilinear maps of rank $\binom{0}{3}$, $\binom{1}{3}$ and $\binom{2}{3}$. An inner product operation between

$\mathbb{T} = \mathbf{e}_\mu T^{\mu\nu} \mathbf{e}_\nu$ and $^{(3)}\Sigma = \mathbf{e}_\sigma \, {}^{(3)}\underset{\sim}{\Sigma}^\sigma \in \binom{1}{3}$ is to be viewed as having an effect only on the coefficient of the 3-form $^{(3)}\underset{\sim}{\Sigma}$ so that

$$^*\mathbb{T} = \mathbb{T} \cdot {}^{(3)}\Sigma = \mathbf{e}_\mu T^{\mu\nu} g_{\nu\sigma} \, {}^{(3)}\underset{\sim}{\Sigma} \in \binom{1}{3}.$$

and similarly

$$^*\mathbb{S} = \mathbb{S} \cdot {}^{(3)}\Sigma = S^\nu g_{\nu\sigma} \, {}^{(3)}\underset{\sim}{\Sigma} \in \binom{0}{3}.$$

Both belong to the space spanned by $\{dx^\alpha \wedge dx^\beta \wedge dx^\gamma\}$, one to the subspace $\binom{1}{3}$, the other to the subspace $\binom{0}{3}$.

Taking the exterior derivative of these 3-forms follows the usual rule $d(f\underset{\sim}{\omega}) = df \wedge \underset{\sim}{\omega} + f\, d\underset{\sim}{\omega}$. Applied to each of the two 3-forms one finds that (10.7)

$$
\begin{aligned}
d(\mathbb{T}\cdot {}^{(3)}\Sigma) &= d\mathbb{T}\cdot \wedge {}^{(3)}\Sigma + \mathbb{T}\cdot d\, {}^{(3)}\Sigma \\
&= \mathbf{e}_\mu T^{\mu\nu}{}_{;\sigma}\, \mathbf{e}_\nu\, dx^\sigma \cdot \wedge {}^{(3)}\Sigma + \mathbb{T}\cdot d\, {}^{(3)}\Sigma \\
&= \mathbf{e}_\mu T^{\mu\nu}{}_{;\sigma}\, \mathbf{e}_\nu \cdot dx^\sigma \wedge \mathbf{e}_\rho\, {}^{(3)}\underset{\sim}{\Sigma}{}^{\rho} + \mathbb{T}\cdot d\, {}^{(3)}\Sigma
\end{aligned}
$$

For the wedge algebra $\mathbf{e}_\rho$ is to be viewed as a mere coefficient. Consequently, it is uneffected by the wedge product operation dx^σ, and one finds that

$$
= \mathbf{e}_\mu T^{\mu\nu}{}_{;\sigma}\, \mathbf{e}_\nu \cdot \mathbf{e}_\rho\, dx^\sigma \wedge {}^{(3)}\underset{\sim}{\Sigma}{}^{\rho} + \mathbb{T}\cdot d\, {}^{(3)}\Sigma
$$

is a vector-valued 4-form, an element of $\binom{1}{4}$. }

Combine Eq. (12.7) with Eq. (12.8), insert the result into Eq. (12.5) and obtain

$$
\begin{aligned}
d({}^*\mathbb{T}) &= \mathbf{e}_\tau \otimes \mathbf{e}_\sigma T^{\tau\sigma}{}_{;\mu} \cdot \mathbf{e}_\nu\, g^{\nu\rho} \sqrt{-g}\, \delta^{\mu}_{\rho}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\
&= \mathbf{e}_\tau T^{\tau\sigma}{}_{;\sigma} \sqrt{-g}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \qquad (12.9)
\end{aligned}
$$

$$
= \begin{pmatrix} \text{amount of momenergy} \\ \text{created in} \\ \text{the invariant spacetime} \\ \text{4-volume } \sqrt{-g}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{pmatrix} \qquad (12.10)
$$

Whenever momenergy is neither created nor (12,8)
destroyed, then such a state of affairs is
mathematized by the statement (12.11)

$$\boxed{T^{\tau\sigma}{}_{;\sigma} = 0} \quad \left(\text{"Conservation of momenergy"}\right)$$