Lecture 13

Stress tensor as the area force relation

In MTW read Chapter 5, starting with Box 5.1 on P131,
then Section § 5.3, p 138

I. Particle-induced Force on an Area. (13.1)

All matter is composed of particles. Their averaged motion and/or interaction across a small area manifests itself as a force on this area. Moreover, for for small areas (but still large enough to preserve the applicability of the averaging process) the relation between the size of the area and force on it is a linear one.

II. Force-area Relation

Focus your mind on a volume element with its bounding surface areas. Each of them has a (spatial) normal vector, and also has a force acting on it. This force vector acting on a surface element characterized by its normal vector is a type of <u>stress</u>. The mathematization process of this circumstance is executed as follows:

A) Elements of Area

Focus your mind on a laboratory coordinate frame coordinatized by (x,y,z) and an element of (triangular) area with vertices at $x=a_1$, $y=a_2$, and $z=a_3$

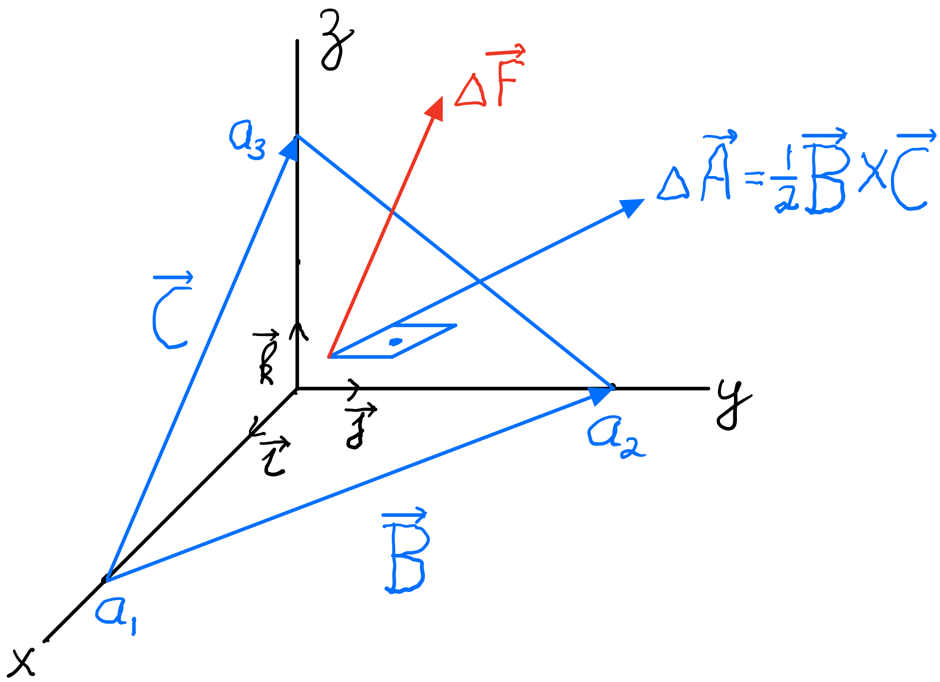

Figure 13.1 Force $\Delta\vec{F}$ acts on area spanned by vectors $\vec{B}$ and $\vec{C}$ and whose normal is $\Delta\vec{A}=\frac{1}{2}\vec{B}\times\vec{C}$.

The area is subtended by the vectors

$$\vec{B} = -a_1\vec{i} + a_2\vec{j}$$
$$\vec{C} = -a_1\vec{i} + a_3\vec{k},$$

and the vector normal to this area is

$$\Delta\vec{A} = \frac{1}{2}\vec{B}\times\vec{C}$$
$$= \frac{1}{2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a_1 & a_2 & 0 \\ -a_1 & 0 & a_3 \end{vmatrix}$$

$$= \frac{1}{2}\left[\vec{i}\, a_2 a_3 + \vec{j}\, a_3 a_1 + \vec{k}\, a_1 a_2\right]$$

This decomposes the normal vector

$$\Delta\vec{A} = \vec{i}\,\Delta A_x + \vec{j}\,\Delta A_y + \vec{k}\,\Delta A_z$$

into its components

$$\Delta A_x = \frac{1}{2} a_2 a_3$$

$$\Delta A_y = \frac{1}{2} a_3 a_1$$

$$\Delta A_z = \frac{1}{2} a_1 a_2$$

relative to the lab basis $(\vec{i}, \vec{j}, \vec{k})$. These lab components are the projections of $\Delta\vec{A}$ onto the respective coordinate planes.

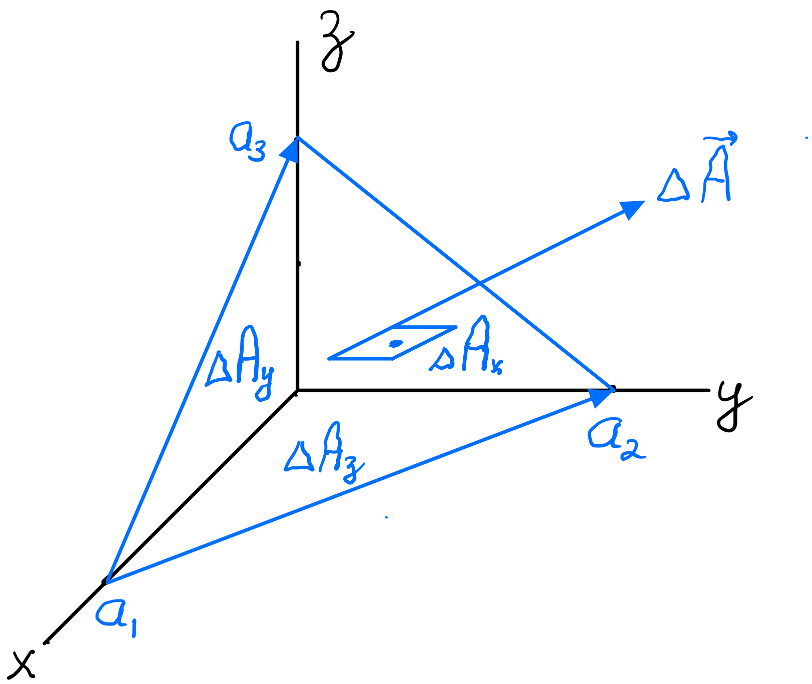

Figure 13.2 Projections of the area, whose normal is $\Delta\vec{A}$, onto the coordinate planes. The sum of the squares of these projections equals the squred magnitude of $\Delta\vec{A}$:

$$(A_x)^2 + (A_y)^2 + (A_z)^2 = \frac{1}{4}\left[a_2^2 a_3^2 + a_3^2 a_1^2 + a_1^2 a_2^2\right] = |\Delta\vec{A}|^2.$$

B.) Stress

The fact that a force field $\vec{F}$ is distributed uniformly over the planar neighborhood which contains $\Delta\vec{A}$ implies that doubling the size of $\Delta\vec{A}$ doubles the size of $\Delta\vec{F}$. In other words, $\Delta\vec{F}$ is a linear function of $\Delta\vec{A}$, i.e. by observing that changing $\Delta\vec{A}$ causes a change in $\Delta\vec{F}$, one says that the causal relationship between is one which is linear.

This linear function is the "<u>stress</u>" to which the matter in the volume element is subjected:

$$\Delta\vec{F} = \vec{i}\,\Delta F^x + \vec{j}\,\Delta F^y + \vec{k}\,\Delta F^z = \text{"stress"}(\Delta\vec{A}),$$

where "stress" is mathematized by the following equations

$$\Delta F^x = T^{xx}\Delta A_x + T^{xy}\Delta A_y + T^{xz}\Delta A_z$$

$$\Delta F^y = T^{yx}\Delta A_x + T^{yy}\Delta A_y + T^{yz}\Delta A_z$$

$$\Delta F^z = T^{zx}\Delta A_x + T^{zy}\Delta A_y + T^{zz}\Delta A_z$$

They comprise the linear causal relationship between vectors $\Delta\vec{A}$ and $\Delta\vec{F}$

Each of the stress components T^{xx}, T^{xy}, etc is measurable. They characterize the stress to which matter in the neighborhood of the origin in Figures 13.1 and 13.2 is subjected to.

For a given volume element of matter these components form a square array

$$\text{"Stress"} = \begin{bmatrix} T^{xx} & T^{xy} & T^{xz} \\ T^{yx} & T^{yy} & T^{yz} \\ T^{zx} & T^{zy} & T^{zz} \end{bmatrix}. \qquad (13.1)$$

Here each of the diagonal elements refers to a pressure,

$$T^{xx} = \frac{(\text{force into x-direction})}{\begin{pmatrix}\text{unit area pointing}\\ \text{into the x-direction}\end{pmatrix}} = \begin{matrix}\text{"pressure}\\ \text{into the}\\ \text{x-direction"},\end{matrix}$$

while each of the off-diagonal elements is a shear stress

$$T^{xy} = \frac{(\text{force into x-direction})}{\begin{pmatrix}\text{unit area pointing}\\ \text{into the y-direction}\end{pmatrix}} = \begin{matrix}\text{"shear}\\ \text{stress"}\end{matrix}$$

In general,

$$T^{ii} = \text{pressure (no sum)}$$

$$T^{ij} = \text{shear stress} \ (i \neq j)$$