LECTURE 14

Decomposition of
The Momenergy Density-Flux

Components of the stress-energy tensor

- Energy density
- Momentum density
- Energy flux
- Momentum flux

(14.1)

The momenergy density-flux

$$^*\mathbb{T} = \mathbf{e}_\mu T^{\mu\nu} \mathbf{e}_\nu \cdot \mathbf{e}_\sigma \epsilon^\sigma{}_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3! \qquad (14.1)$$

is the product of a mental integration of

momenergy $\qquad \mathbf{p} = \mathbf{e}_\mu p^\mu,$

the particle 4-current

$$\mathbf{S} = N u^\nu \mathbf{e}_\nu ,$$

and the vector-valued measure*

$$^{(3)}\Sigma = \mathbf{e}_\sigma \overset{(3)}{\Sigma} \equiv \mathbf{e}_\sigma \epsilon^\sigma{}_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3!,$$

which is a 3-form with vectorial coefficients.**

*\footnote{This vector-valued measure was introduced en passant by MTW with their Eq.(15.15) in order to mathematize the "moment of rotation" of the Einstein field equations.

Modulo a minus sign, they introduce this "measure" by

$$\star(A\wedge B\wedge C) \equiv \mathbf{e}_\sigma \epsilon^\sigma{}_{\alpha\beta\gamma} A^\alpha B^\beta C^\gamma$$
$$= \mathbf{e}_\sigma \epsilon^\sigma{}_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3!\,(A,B,C)$$

This is a direct generalization of the familiar cross product of a pair of vectors $\vec{A} = A^j \frac{\partial}{\partial x^j}$ and $\vec{B} = B^k \frac{\partial}{\partial x^k}$ in 3-d Euclidean space:

This vector

$$\star(\vec{A}\wedge\vec{B}) = \mathbf{e}_\ell \epsilon^\ell{}_{jk} A^j B^k$$
$$= \mathbf{e}_\ell \epsilon^\ell{}_{jk}\, dx^j \wedge dx^k/2!\,(\vec{A},\vec{B})$$
$$\equiv {}^{(2)}\Sigma(\vec{A},\vec{B})$$

in Euclidean

This is because the curvilinear coordinate basis expansion of the cross product is

$$\left(\vec{A}\times\vec{B}\right)^{\ell}\frac{\partial}{\partial x^{\ell}} = e_{\ell}\ \frac{\partial(x,y,z)}{\partial(x^1,x^2,x^3)}\begin{vmatrix} g^{\ell 1} & g^{\ell 2} & g^{\ell 3} \\ A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \end{vmatrix}$$

$$= e_{\ell}\, g^{\ell i}\, \epsilon_{ijk}\, A^j B^k$$

$$= e_i\, \epsilon^{i}{}_{jk}\, dx^j \wedge dx^k\, (\vec{A},\vec{B})$$

$$= \bigstar\, (\vec{A},\vec{B}) \qquad \}$$

**\footnote{ As compared to a scalar-valued 3-form, such as $^*S = S^{\nu}\epsilon_{\nu\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}/3!$, which was identified in Lecture 8 by Eq. (8.3), which is a scalar valued measure, and which measures the number of particles in the 3-d volume elements spanned by some triad of four-dimensional vectors (A,B,C),

$$\# = {}^*S^{\mu}\,(A,B,C)$$

$$= S^{\nu}\epsilon_{\nu\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}/3!\ (A,B,C)$$

$$= \sqrt{-g}\begin{vmatrix} S^0 & S^1 & S^2 & S^3 \\ A^0 & A^1 & A^2 & A^3 \\ B^0 & B^1 & B^2 & B^3 \\ C^0 & C^1 & C^2 & C^3 \end{vmatrix} \quad . \qquad \}$$

Applied to matter composed of a variety of particle species, this product is an organic whole each of whose three factors is a spacetime coordinate frame invariant. Its explicit form is

$$^*\mathbb{T} = e_{\mu}\sum_{a} p^{\mu} N_a u_a^{\nu}\, e_{\nu}\cdot e_{\sigma}\, \epsilon^{\sigma}{}_{\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}/3! \qquad (14.2)$$

This spacetime invariant geometrizes four measurable physics concepts: energy density,

energy flux, momentum density, and momentum flux. All of them are packed into $^*\mathbb{T}$. Their measurability requires their quantitative identification by unpacking $^*\mathbb{T}$ so as to exhibit them explicitly.
This unpacking process consists of subjecting $^*\mathbb{T}$ in Eq. (14.1) a 3+1, i.e. space plus time, decomposition.

The explicit split of $^*\mathbb{T}$ into its space and time components is a matter isolating the time and the spatial terms of the double sum in Eq. (14.1):

$$\mathbb{T} = e_\mu T^{\mu\nu} e_\nu = \left(e_0 T^{00} + e_\ell T^{\ell 0}\right) e_0 + \left(e_0 T^{0m} + e_\ell T^{\ell m}\right) e_m \quad (14.3)$$

Here the latin summation indeces range only over the spatial indeces 1, 2, and 3.
Similarly one has

$$\begin{aligned} {}^{(3)}\Sigma &= e_\sigma \, \epsilon^\sigma{}_{\alpha\beta\gamma} \, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3! \\ &= \left(e_0 \, \epsilon^0{}_{\alpha\beta\gamma} + e_\ell \, \epsilon^\ell{}_{\alpha\beta\gamma}\right) dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3! \end{aligned} \quad (14.4)$$

Applying these decompositions to Eqs. (14.1) and (14.2) decomposes both the geometrical and the physical mathematization of $^*T$: (14.4)

$$^*\mathbb{T} = \left(\mathbf{e}_0 T^{00} + \mathbf{e}_m T^{m0}\right)\epsilon_{0\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3!$$
$$+\left(\mathbf{e}_0 T^{0\ell} + \mathbf{e}_m T^{m\ell}\right)\epsilon_{\ell\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3! \qquad (14.5)$$

and

$$^*\mathbb{T} = \left(\mathbf{e}_0 \sum_a p^0 N_a u_a^0 + \mathbf{e}_m \sum_a p^m N_a u_a^0\right)\epsilon_{0\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3! +$$
$$+\left(\mathbf{e}_0 \sum_a p^0 N_a u_a^\ell + \mathbf{e}_m \sum_a p^m N_a u_a^\ell\right)\epsilon_{\ell\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3! \qquad (14.6)$$

1.) Density Components of $\mathbb{T}$.

The physical meaning of T^{00} and T^{m0} are recovered from $^*T$ by evaluating it on the triad of space-like vectors

$$A = \Delta x \frac{\partial}{\partial x^1}, \quad B = \Delta y \frac{\partial}{\partial x^2}, \quad C = \Delta z \frac{\partial}{\partial x^3}.$$

From Eqs. (14.5) and (14.6) one finds that

$$\left(\mathbf{e}_0 T^{00} + \mathbf{e}_m T^{m0}\right)\epsilon_{0123}\,\Delta x \Delta y \Delta z = {}^*\mathbb{T}(A,B,C) = \left(\mathbf{e}_0 \sum_a p^0 N_a u_a^0 + \mathbf{e}_m \sum_a p^m N_a u_a^0\right)\epsilon_{0123}\,\Delta x \Delta y \Delta z$$

a) The fact that

$$\sum_a p_a^0 N_a u_a^0\, \Delta x \Delta y \Delta z = \begin{pmatrix}\text{mass-energy} \\ \text{observed in} \\ \text{volume } \Delta x \Delta y \Delta z\end{pmatrix}$$

implies

$$\boxed{T^{00} = \frac{\{\text{mass-energy}\}}{(\text{volume})} = \text{"energy density"}.}$$

b) The fact that

$$\sum_a P_a^m N_a U_a^0 \,\Delta x \Delta y \Delta z = \begin{pmatrix} \text{mass-energy} \\ \text{observed in} \\ \text{volume } \Delta x \Delta y \Delta z \end{pmatrix}$$

implies

$$\boxed{e_m T^{m0} = \frac{\{\overrightarrow{\text{momentum}}\}}{(\text{volume})} = \text{"momentum density"}}$$

2.) Flux Components of $\mathbb{T}$.

The physical meaning of $T^{0\ell}$ and $T^{m\ell}$ are recovered from ${}^*\mathbb{T}$ by evaluating it on the triad of vectors one of which is time-like

$$A = -\Delta t \frac{\partial}{\partial x^0}\,,\quad B = \Delta y \frac{\partial}{\partial x^2}\,,\quad C = \Delta z \frac{\partial}{\partial x^3}.$$

From Eqs. (14.5) and (14.6) one finds that

$$(e_0 T^{01} + e_m T^{m1})\,\epsilon_{1023}(-)\,\Delta t \Delta y \Delta z = {}^*\mathbb{T}(A,B,C) = \Big(e_0 \sum_a p^0 N_a u_a^1 + e_m \sum_a p^m N_a u_a^1\Big)\,\epsilon_{1023}(-)\,\Delta t \Delta y \Delta z$$

a) The fact that

$$\sum_a P_a^0 N_a U_a^1 \,\Delta t \Delta y \Delta z = \left(\begin{array}{l} \text{mass-energy} \\ \text{observed to pass} \\ \text{through area } \Delta y \Delta z \\ \text{during time } \Delta t \end{array} \right)$$

implies that

$$\boxed{T^{01} = \frac{(\text{mass-energy})}{(\text{time})(\overrightarrow{\text{area}})^x} = \text{"energy flux into the x-direction"} = \text{"power per x-directed area"} = \text{"intensity"}}$$

b) The fact that

$$e_m \sum_a P_a^m N_a U_a \,\Delta t \Delta y \Delta z = \left(\begin{array}{l} \{\overrightarrow{\text{momentum}}\} \\ \text{observed to pass} \\ \text{through area } \Delta y \Delta z \\ \text{during time } \Delta t \end{array} \right)$$

implies

$$e_m T^{m1} = \frac{\overrightarrow{\text{momentum}}}{(\text{time})(\overrightarrow{\text{area}})^x} = \text{"Force per x-directed area"}$$

In particular,

T^{11} = pressure along the x-direction

T^{21} = shear stress:

= "y-force per x-area"

= "y-momentum flux into x-direction"

3. Summary.

By observing matter, i.e. particles in aggregate characterized by p, N, and u, one finds that the momenergy 4-current

$$\mathbb{T} = \mathbf{e}_\mu T^{\mu\nu} \mathbf{e}_\nu$$

has components that decompose into

$$[T^{\mu\nu}] = \begin{array}{cc} \left[\begin{array}{c|c} T^{00} & T^{0\ell} \\ \hline T^{m0} & T^{m\ell} \end{array}\right] & \begin{array}{l} \}\ \text{energy} \\ \}\ \text{momentum} \end{array} \\ \begin{array}{cc} \underbrace{\qquad}_{\text{density}} & \underbrace{\qquad\qquad}_{\text{flux}} \end{array} & \end{array}$$