Lecture 15

Conservation of Momenergy

In MTW read (i.e. grasp) § 5.4 and Box 5.4

In Wheeler's A JOURNEY INTO GRAVITY AND SPACETIME read Chapter 6 on Momenergy.
This <u>very</u> readable chapter particularizes and conceptualizes the phyical basis of momenergy and its conservation.

Momenergy is conserved not only on the retail level in isolated collision processes (Lecture 7) here and there but also on the wholesale level of trillions of collision in the Mike hydrogen bomb explosion of Novembe 1, 1952, in the barrel of matter at the center of a star, or in the kettle of boiling water.

I. Change in volume content vs. outflow across its faces.

Momenergy conservation is the statement that momenergy is neither created nor destroyed.

Every measurement is based on a standard.

For momenergy this consists of a triad of displacement vectors, say A, B, and C, in spacetime.

If they all are space-like then one has a spatial volume $\Delta x \Delta y \Delta z$ as the standard.

If one of them is time-like then one has each of the three temporal volumes $\Delta t \Delta y \Delta z$, $\Delta t \Delta z \Delta x$, and $\Delta t \Delta x \Delta y$ as the standard; which one depends on which of the areas $\Delta y \Delta z$, $\Delta z \Delta x$, or $\Delta x \Delta y$ are experimentally relevant.

The statement of momenergy conservation necessitates all four of the spacetime volumes.

A cubical domain of volume $\Delta x \Delta y \Delta z$ during the time

interval Δt sweeps out the 4-d spacetime volume $\Delta t\Delta x\Delta y\Delta z$.
If there is any momenergy created in $\Delta t\Delta x\Delta y\Delta z$, then this is a unique index, a two-part sum of momenergy, each part measurable and accountable and given by the vectorial index

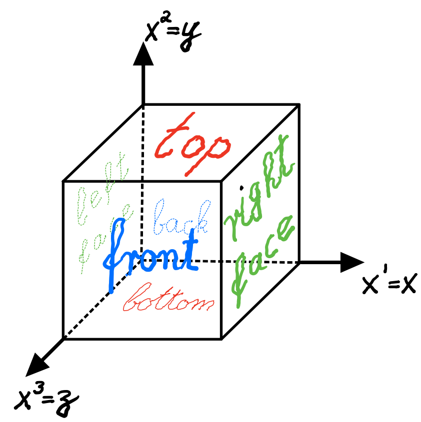
$$Q = \begin{pmatrix}\text{momenergy}\\ \text{created in } \Delta t\Delta x\Delta y\Delta z\end{pmatrix}$$

$$= \begin{pmatrix}\text{change in m.e.}\\ \text{inside } \Delta x\Delta y\Delta z\\ \text{during time } \Delta t\end{pmatrix} + \begin{pmatrix}\text{outflow of m.e.}\\ \text{through the sides}\\ \text{of } \Delta x\Delta y\Delta z \text{ during}\end{pmatrix};$$

the explicit form of these two contributions

$$= \begin{pmatrix}\text{m.e. in } \Delta x\Delta y\Delta z\\ \text{at the end, } t+\Delta t,\\ \text{of time interval } \Delta t\end{pmatrix} - \begin{pmatrix}\text{m.e. in } \Delta x\Delta y\Delta z\\ \text{at the beginning, } t,\\ \text{of time interval } \Delta t\end{pmatrix}$$

$$\left.\begin{array}{l}
+ \begin{pmatrix}\text{flow of m.e. out}\\ \text{of right hand}\\ \text{face of } \Delta x\Delta y\Delta z\\ \text{during } \Delta t\end{pmatrix} + \begin{pmatrix}\text{flow of m.e. out}\\ \text{of left hand}\\ \text{face of } \Delta x\Delta y\Delta z\\ \text{during } \Delta t\end{pmatrix}\\
+ \begin{pmatrix}\text{flow of m.e. out}\\ \text{of top}\\ \text{face of } \Delta x\Delta y\Delta z\\ \text{during } \Delta t\end{pmatrix} + \begin{pmatrix}\text{flow of m.e. out}\\ \text{of bottom}\\ \text{face of } \Delta x\Delta y\Delta z\\ \text{during } \Delta t\end{pmatrix}\\
+ \begin{pmatrix}\text{flow of m.e. out}\\ \text{of front}\\ \text{face of } \Delta x\Delta y\Delta z\\ \text{during } \Delta t\end{pmatrix} + \begin{pmatrix}\text{flow of m.e. out}\\ \text{of back}\\ \text{face of } \Delta x\Delta y\Delta z\\ \text{during } \Delta t\end{pmatrix}
\end{array}\right\}$$



$$\boxed{\text{Conservation of momenergy: } Q=0}$$

Example

1. $\begin{pmatrix} \text{change in m.e.} \\ \text{in } \Delta x \Delta y \Delta z \end{pmatrix} < 0 \iff \underline{\text{net outflow}} > 0$

2. $\begin{pmatrix} \text{change in m.e.} \\ \text{in } \Delta x \Delta y \Delta z \end{pmatrix} > 0 \iff \underbrace{\underline{\text{net outflow}} < 0}_{\text{"inflow"}}$

3. $\begin{pmatrix} \text{change in m.e.} \\ \text{in } \Delta x \Delta y \Delta z \end{pmatrix} = 0 \iff \underline{\text{net outflow}} = 0$

II. Mathematization of change in volume content and of outward flow.

The descriptive momenergy ledger for Q calls for four triads of 4-d vectors. One triad for specifying the interior of the volume element $\Delta x \Delta y \Delta z$ to accommodate the measured density of momenergy, and the other three for specifying the time-like domains $\Delta t \Delta y \Delta z$, $\Delta t \Delta z \Delta x$, and $\Delta t \Delta x \Delta y$ to accommodate the measured flux of momenergy.*

* \footnote{ Recall that

these measured momenergy quantities are condensed (15.4)

into the momenergy density-flux (See Lecture 12)

$$ {}^{*}\mathbb{T} = \mathbb{e}_\mu T^{\mu\nu} \mathbb{e}_\nu \cdot \mathbb{e}_\sigma\, \epsilon^{\sigma}{}_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma / 3! \qquad (15.1) $$

where

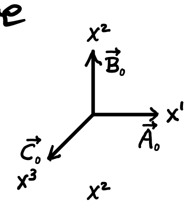
$$ [T^{\mu\nu}] = \begin{array}{cl} \left[\begin{array}{c|c} T^{00} & T^{0\ell} \\ \hline T^{m0} & T^{m\ell} \end{array}\right] & \begin{array}{l} \}\ \text{energy} \\ \}\ \text{momentum} \end{array} \\ \begin{array}{cc} \underbrace{\quad}_{\text{density}} & \underbrace{\qquad\quad}_{\text{flux}} \end{array} & \end{array} $$

are the coefficients of its momenergy 4-current

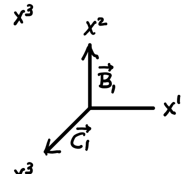
$$ \mathbb{T} = \mathbb{e}_\mu T^{\mu\nu} \mathbb{e}_\nu \,. $$

These four triads of vectors are

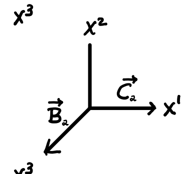
$$ A_0 = \Delta x \frac{\partial}{\partial x^1}, \quad B_0 = \Delta y \frac{\partial}{\partial x^2}, \quad C_0 = \Delta z \frac{\partial}{\partial x^3} $$



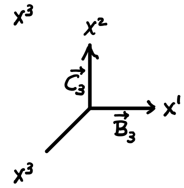
$$ A_1 = \Delta t \frac{\partial}{\partial x^0}, \quad B_1 = \Delta y \frac{\partial}{\partial x^2}, \quad C_1 = \Delta z \frac{\partial}{\partial x^3} $$



$$ A_2 = \Delta t \frac{\partial}{\partial x^0}, \quad B_2 = \Delta z \frac{\partial}{\partial x^3}, \quad C_2 = \Delta x \frac{\partial}{\partial x^1} $$



$$ A_3 = \Delta t \frac{\partial}{\partial x^0}, \quad B_3 = \Delta x \frac{\partial}{\partial x^1}, \quad C_3 = \Delta y \frac{\partial}{\partial x^2} $$



The momenergy density-flux 3-form $^*T$ evaluated * on these vectors quantifies the contribution to the Q index of momenergy created by the volume element's interior $\Delta x \Delta y \Delta z$ during the time interval Δt.

The result is that the momenergy ledger consist of four paired ** contributions,

$$
\begin{aligned}
Q = {} & e_\mu T^{\mu 0}\sqrt{-g}\,\Big|_{x^0+\Delta t} \Delta x \Delta y \Delta z - e_\mu T^{\mu 0}\sqrt{-g}\,\Big|_{x^0} \Delta x \Delta y \Delta z \\
& + e_\mu T^{\mu 1}\sqrt{-g}\,\Big|_{x^1+\Delta x} \Delta y \Delta z \Delta t + e_\mu T^{\mu 1}\sqrt{-g}\,\Big|_{x^1} (-)\Delta y \Delta z \Delta t \\
& + e_\mu T^{\mu 2}\sqrt{-g}\,\Big|_{x^2+\Delta y} \Delta z \Delta x \Delta t + e_\mu T^{\mu 2}\sqrt{-g}\,\Big|_{x^2} (-)\Delta z \Delta x \Delta t \\
& + e_\mu T^{\mu 3}\sqrt{-g}\,\Big|_{x^3+\Delta z} \Delta x \Delta y \Delta t + e_\mu T^{\mu 3}\sqrt{-g}\,\Big|_{x^3} (-)\Delta x \Delta y \Delta t
\end{aligned}
$$

*\footnote{ for p 15,5

The evaluation of ${}^*\mathbb{T}$ on each of the triads of vectors follows the same computational line of reasoning. Thus,

$$
\begin{aligned}
{}^*\mathbb{T}(A_0, B_0, C_0) &= e_\mu T^{\mu\nu} e_\nu \cdot e_\sigma \epsilon^\sigma{}_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3!\,(A_0, B_0, C_0)\\
&= e_\mu T^{\mu\nu} \epsilon_{\nu\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3!\,(A_0, B_0, C_0)\\
&= e_\mu T^{\mu\nu} \epsilon_{\nu 123}\, \Delta x\, \Delta y\, \Delta z\\
&= e_\mu T^{\mu 0} \sqrt{-g}\, \Delta x\, \Delta y\, \Delta z,
\end{aligned}
$$

$$
\begin{aligned}
{}^*\mathbb{T}(A_1, B_1, C_1) &= e_\mu T^{\mu\nu} \epsilon_{\nu\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3!\,(A_1, B_1, C_1)\\
&= e_\mu T^{\mu\nu} \epsilon_{\nu 023} (-) \Delta t\, \Delta y\, \Delta z\\
&= e_\mu T^{\mu 1} \epsilon_{1023} (-) \Delta t\, \Delta y\, \Delta z\\
&= e_\mu T^{\mu 1} \sqrt{-g}\, \Delta t\, \Delta y\, \Delta z,
\end{aligned}
$$

and similarly for the others. }

**\footnote{ The all-important minus sign in second contribution for the volume elements $\Delta t\, \Delta y\, \Delta z$, $\Delta t\, \Delta z\, \Delta x$, and $\Delta t\, \Delta x\, \Delta y$ is due to the fact that outward normal points into the direction opposite to that of the first contribution. }

or, neglecting higher order terms,

$$Q = \nabla_0\left(e_\mu T^{\mu 0}\sqrt{-g}\right)\Delta t\Delta x\Delta y\Delta z$$
$$+\nabla_1\left(e_\mu T^{\mu 1}\sqrt{-g}\right)\Delta x\Delta y\Delta z\Delta t$$
$$+\nabla_2\left(e_\mu T^{\mu 2}\sqrt{-g}\right)\Delta y\Delta z\Delta x\Delta t$$
$$+\nabla_3\left(e_\mu T^{\mu 3}\sqrt{-g}\right)\Delta z\Delta x\Delta y\Delta t$$
$$= \nabla_\nu\left(e_\mu T^{\mu\nu}\sqrt{-g}\right)\Delta t\Delta x\Delta y\Delta z$$

Based the properties of the covariant derivative operator ∇ this expressed becomes

$$= e_\mu T^{\mu\nu}{}_{;\nu}\sqrt{-g}\; dx^0\wedge dx^1\wedge dx^2\wedge dx^3\,(A_1, A_0, B_0, C_0)$$
$$= d({}^*\mathbb{T})(A_1, A_0, B_0, C_0)$$

where $d({}^*\mathbb{T})$ is the exterior derivative of

$${}^*\mathbb{T} = e_\mu T^{\mu\nu}\epsilon_{\nu\alpha\beta\gamma}\, dx^\alpha\wedge dx^\beta\wedge dx^\gamma/3!$$

Relative to the rectilinear parallel (t,x,y,z)-induced basis one has $\nabla_\nu e_\nu = 0$. Consequently, the created momenergy/4-volume is

$$\frac{Q}{\Delta t\Delta x\Delta y\Delta z} = e_\mu T^{\mu\nu}{}_{,\nu}$$

Relative to curvilinear coordinates this momenergy is

$$\frac{Q}{\sqrt{-g}\,\Delta x^0 \Delta x^1 \Delta x^2 \Delta x^3} = \mathbf{e}_\mu T^{\mu\nu}{}_{;\nu}$$

III. Conservation of momenergy

However, in nature momenergy is neither created nor destroyed:

$$Q = 0.$$

The local mathematization of this fact is the statement

$$\boxed{\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0} \quad \mu = 0,1,2,3$$

or more generally

$$\boxed{T^{\mu\nu}{}_{;\nu} = 0} \quad \mu = 0,1,2,3 \,,$$

which is equivalent to

$$\boxed{d\,{}^*\mathbb{T} = 0}\,.$$