Lecture 16

Relativistic Fluid Dynamics

I. Momenergy Tensor of a Perfect Fluid

II. Momenergy of an Electromagnetic Field

III. Conservation of Momenergy

In MTW read and peruse

Ex. 22.1 : Change in comoving volume : Kinematics

§ 22.5 : Matter in motion : Dynamics

Box 5.5 : Momenergy for a perfect fluid

Ex. 3.18 : Momenergy for E,&M

Page 152,155 : Conservation momenergy of charged matter plus e.m. radiation

I. Momenergy tensor of a Perfect Fluid (16.1)

In its comoving coordinate frame, where its spatial velocity is zero, a perfect fluid is characterized by its three attributes:

- mass-energy density $\rho(x^\alpha)$
- isotropic pressure $p(x^\alpha)$
- 4-velocity with components $\{u^\mu(x^\alpha)\} = \{1, 0, 0, 0\}$

The momenergy tensor components relative to such a frame are

$$[T^{\mu\nu}] = \begin{bmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & & \\ & p & & \\ & & p & \\ & & & p \end{bmatrix}$$

$$[T_\mu{}^\nu] = \rho \begin{bmatrix} -1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} + p \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} =$$

$$= \left[\rho \quad u_\mu u^\nu \quad + p\,(\delta_\mu{}^\nu + u_\mu u^\nu)\right]$$

$$T_A{}^B = \rho\, u_A u^B + p\left(\delta_A{}^B + u_A u^B\right)$$

$$T_a{}^b = p_\perp \delta_a^b$$

Relative to any frame the momenergy tensor components for a perfect fluid are therefore

$$\boxed{T_\mu{}^\nu = (p+\rho)\, u_\mu u^\nu + p\, \delta_\mu{}^\nu} \qquad (16.1)$$

(16.2)

II. Momenergy of an Electromagnetic Field

The electromagnetic field

$$\underset{\sim}{F} = F_{\mu\nu}\, dx^\mu \wedge dx^\nu / 2!$$

has a momenergy tensor whose components are

$$T^{\mu\nu}_{e.m.} = \frac{1}{4\pi} F^{\mu\alpha} F^{\nu}{}_{\alpha} - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \qquad (16.2)$$

The 4-current of a charged fluid with comoving particle density N is a 4-vector whose components are

$$J^\mu = qNu^\mu. \qquad (16.3)$$

Here q is the charge per particle so that qN is the comoving charge density.

It is an exercise in "index gymnastic" (MTW, Ex. 3.18, p89) to show that, in light of the Maxwell field equations, the divergence of the e.m. momenergy tensor has components given by

$$T^{\mu\nu}_{e.m.;\nu} = -F^{\mu\alpha} J_\alpha \qquad (16.4)$$

III. Conservation of Momenergy

It is a fact that total momenergy, that of matter plus that of the e.m. field, is neither created nor destroyed. This is mathematized by the statement

$$(T^{\mu\nu}_{matter} + T^{\mu\nu}_{e.m.})_{;\nu} = 0,$$

16.3

or in light of Eqs. (16.3)-(16.4)

$$\boxed{T^{\mu\nu}_{\;\;matter;\nu} = -NqF^{\mu}{}_{\alpha}u^{\alpha}} \qquad (16.5)$$

The right hand side mathematizes the amount of momenergy injected into the invariant 4-d spacetime volume $\Delta\tau\, v$.

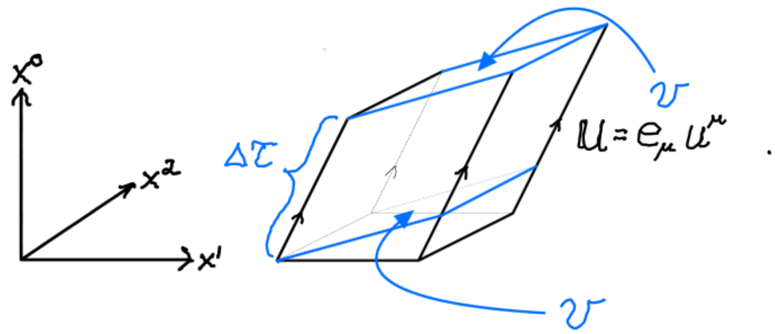

Figure 16.1 : World tube bounded by and filled with congruent particle world lines.
The 4-d basis invariant spacetime volume $\Delta\tau v$ is the product of the elapsed proper time $\Delta\tau$ in a comoving element of fluid whose comoving volume is v.

Indeed, one has

$$N \cdot qF^{\mu}{}_{\alpha}u^{\alpha} = \frac{(\text{particles})}{\begin{pmatrix}\text{comoving}\\ \text{volume}\end{pmatrix}} \cdot \frac{\begin{pmatrix}\text{Lorentz}\\ \text{4-force}\end{pmatrix}^{\mu}}{(\text{particle})}.$$

But recall that

$$qF^{\mu}{}_{\alpha}u^{\alpha} = \frac{(\Delta\,\text{momenergy})^{\mu}}{(\Delta\tau)} \approx \frac{dp^{\mu}}{d\tau}$$

and

$$N = \frac{\#}{v}$$

16.4

Consequently,

$$N \cdot qF^{\mu}{}_{\alpha} u^{\alpha} = \frac{\#}{v} \times \frac{(\Delta \text{momenergy})^{\mu}}{(\Delta\tau)}$$

$$= \frac{\Delta(\text{total momenergy})}{v\Delta\tau}$$

$$= \frac{(\text{momenergy})^{\mu}}{\begin{pmatrix}\text{invariant}\\ \text{spacetime}\\ \text{volume}\end{pmatrix}}$$

Thus Eq. (16.5) mathematizes the amount of mechanical momenergy created per 4-volume $\Delta\tau\, v$ by the e.m. field $F^{\mu}{}_{\alpha}$ interacting with the fluid particles each one of which has charge q.

Q: what is the mechanical dynamics of the responding particles?