Lecture 17

Relativistic Hydrodynamics

In MTW read § 22.2 and § 22.3

For our purposes one may initially set the entropy in these sections equal to zero

(17.1)

The power of the mathematatized law of momenergy conservation, when applied to the mechanics of a continuum, is that it furnishes the dynamical equations that govern it. For a perfect fluid whose momenergy tensor, whose components are

$$T^{\mu\nu} = (p+\rho)u^\mu u^\nu + g^{\mu\nu}p, \qquad (17.1)$$

these equation are the relativistic equations of motion for a fluid:

$$0 = T^{\sigma\nu}{}_{;\nu} = \left(g^{\sigma\mu}T_\mu{}^\nu\right)_{;\nu} = \underbrace{g^{\sigma\mu}{}_{;\nu}}_{\text{zero}}T_\mu{}^\nu + g^{\sigma\mu}T_\mu{}^\nu{}_{;\nu}$$

or

$$0 = T_\mu{}^\nu{}_{;\nu} = \left[(p+\rho)u_\mu u^\nu + p\,\delta_\mu{}^\nu\right]_{;\nu} \qquad (17.2)$$

$$= \left[(p+\rho)u_\mu\right]_{;\nu}u^\nu + (p+\rho)u_\mu u^\nu{}_{;\nu} + p_{,\mu} \qquad (17.3)$$

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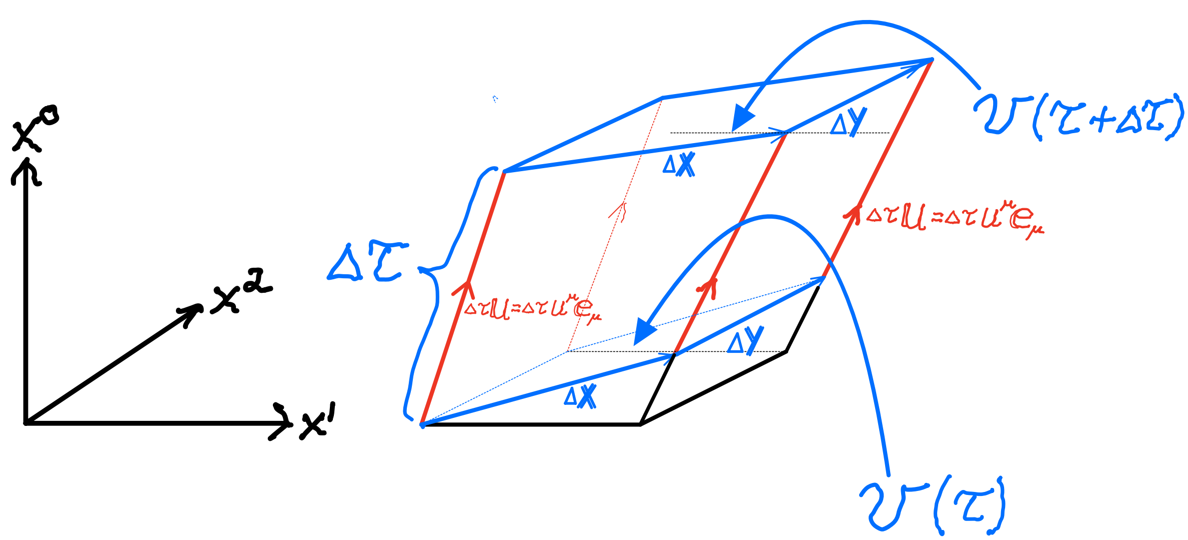


Figure 17.1 World tube whose boundary is generated by the world lines of adjacent particles, each one having its own 4-velocity $\mathbf{u} = \mathbf{e}_\mu u^\mu$. When the 4-velocities of these particles diverge, the comoving volume $\mathcal{V}$ bounded by these particles increases. Over the proper time interval $[\tau, \tau + \Delta\tau]$ this increase is

$$\Delta \mathcal{V} = \mathcal{V}(\tau + \Delta\tau) - \mathcal{V}(\tau),$$

a difference which depends on the divergent nature of the particle 4-velocity vector field $\mathbf{u} = \mathbf{e}_\mu u^\mu$.

This difference is mathematized by the following

Theorem.

GIVEN: A triad of 4-d spatial comoving displacement vectors

$$\Delta \mathbf{X} = \Delta x \frac{\partial}{\partial x} = \Delta X^\alpha \frac{\partial}{\partial x^\alpha}$$

$$\Delta \mathbf{Y} = \Delta y \frac{\partial}{\partial y} = \Delta Y^\beta \frac{\partial}{\partial x^\beta}$$

$$\Delta \mathbf{Z} = \Delta z \frac{\partial}{\partial z} = \Delta Z^\gamma \frac{\partial}{\partial x^\gamma}$$

emanating from a common event on the world line of a reference particle whose 4-velocity is

$$\mathbf{U} = \frac{\partial}{\partial \tau} = u^\mu \frac{\partial}{\partial x^\mu}$$

Here $\{\frac{\partial}{\partial \tau}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ is the comoving orthonormal basis, while $\{\frac{\partial}{\partial x^\mu} = e_\mu\}_{\mu=0}^{3}$ is a generic coordinate-induced basis.

CONCLUSION

1.) The comoving 3-volume v spanned by this triad is

$$\begin{aligned} v &= u^\mu \epsilon_{\mu\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma / 3!\,(\Delta X, \Delta Y, \Delta Z) \\ &= u^\mu \epsilon_{\mu\alpha\beta\gamma}\, \Delta X^\alpha \Delta Y^\beta \Delta Z^\gamma \\ &= \Delta x\, \Delta y\, \Delta z\,. \end{aligned}$$

2.) The increase in this comoving volume during $[\tau, \tau+\Delta\tau]$ is

$$v(\tau+\Delta\tau) - v(\tau) \equiv \Delta v = v\, u^\mu{}_{;\mu}\, \Delta\tau,$$

i.e.

$$\boxed{u^\mu{}_{;\mu} = \frac{1}{v}\frac{dv}{d\tau}} \qquad (17.4)$$

The divergence of the fluid 4-velocity equals the fractional rate of change of a comoving element of fluid volume.

The mathematical proof of this kinematic law of motion is consigned to the appendix of this Lecture 17.

A. Relativistic generalization of Newton's 2nd Law

Introduce the kinematic law Eq. (17.4) into Eq.(17.2).

Recall that v is a scalar function on spacetime and therefore $$\frac{dv}{d\tau} = v_{;\alpha}\frac{dx^\alpha}{d\tau}.$$ (17.3)

Consequently, the resulting form of the conservation law,

$$0 = v\,T_\mu{}^\nu{}_{;\nu} = v\left[(p+\rho)u_\mu\right]_{;\nu}\frac{dx^\nu}{d\tau} + (p+\rho)u_\mu\frac{dv}{d\tau} + v\,p_{;\mu}$$

simplifies into

$$\left[v(p+\rho)u_\mu\right]_{;\nu}\frac{dx^\nu}{d\tau} = -v\,p_{;\mu} \quad . \qquad (17.5)$$

The left hand side is the convective time derivative

$$\frac{D}{d\tau} \equiv [\quad]_{;\nu}\frac{dx^\nu}{d\tau}$$

of the relativistic momenergy in the 3-volume v. The right hand side is the relativistic buoyancy force; it is a volume force whose cause is the pressure gradient.

Because of these identifications,

$$\frac{D}{d\tau}\left[v(p+\rho)u_\mu\right] = -v\frac{\partial p}{\partial x^\mu} \qquad \mu = 0,1,2,3$$

mathematizes the fact that the rate of change of fluid 4-momentum in a comoving volume is controlled by the 4-d pressure gradient acting on this volume.

B. Energy conservation

Momenergy conservation is the statement that in any comoving volume $\mathcal{V}$ during its proper time interval $\Delta\tau$ there is no creation or annihilation of momenergy. The vectorial index of momenergy creation per spacetime 4-volume,

$$\frac{Q}{(\Delta\tau\,\mathcal{V})} = \mathbf{e}_\mu\, T^{\mu\nu}{}_{;\nu} = \frac{(\text{momenergy})}{\begin{pmatrix}\text{invariant}\\ \text{spacetime}\\ \text{4-volume}\end{pmatrix}}$$

vanishes everywhere at all times! This applies also to the energy observed in the comoving frame. Thus,

$$\mathbf{u}\cdot\mathbf{e}_\mu\, T^{\mu\nu}{}_{;\nu} = u^\mu\, T_\mu{}^\nu{}_{;\nu} = 0.$$

Apply this to Eq. (17.2) on page 17.1 and find

$$0 = u^\mu T_\mu{}^\nu{}_{;\nu} = \underbrace{u^\mu (\cancel{p}+\rho)_{,\nu}\, u_\mu}_{=-1} u^\nu + \underbrace{u^\mu (p+\rho)\, u_{\mu;\nu}}_{=0} u^\nu + \underbrace{u^\mu (p+\rho)\, u_\mu}_{=-1} \underbrace{u^\nu{}_{;\nu}}_{\frac{1}{\mathcal{V}}\frac{d\mathcal{V}}{d\tau}} + u^\mu \cancel{p}_{,\mu}$$

$$= -\frac{d\rho}{d\tau} - (p+\rho)\frac{1}{\mathcal{V}}\frac{d\mathcal{V}}{d\tau}$$

Thus $\quad \mathcal{V}\dfrac{d\rho}{d\tau} + \rho\dfrac{d\mathcal{V}}{d\tau} = -p\dfrac{d\mathcal{V}}{d\tau}$

or

$$\boxed{d(\rho\mathcal{V}) = -p\,d\mathcal{V}} \qquad (17.6)$$

This holds along every comoving volume element v.
This equation mathematizes

$$\begin{pmatrix}\text{change in energy}\\ \text{in volume}\end{pmatrix} = \begin{pmatrix}\text{compressional mechanical}\\ \text{work done on the fluid}\\ \text{volume element}\end{pmatrix}$$

This is the 1st law of thermodynamics.

C. Chemical potential

The concept of a chemical potential (a.k.a. enthalpy, a.k.a. injection energy) arises whenever matter exhibits particle conservation, i.e. when the number of particles in a comoving element of volume is a constant,

$$\frac{d(Nv)}{d\tau} = 0. \quad (\text{"particle conservation"})$$

Here

$$N = \frac{(\#\text{ of particles})}{\begin{pmatrix}\text{"comoving}\\ \text{volume"}\end{pmatrix}}$$

This law is geometrized in terms of the particle 4-velocity by observing that

$$0 = v\frac{dN}{d\tau} + N\frac{dv}{d\tau}$$

$$= v N_{,\nu} u^\nu + N u^\nu{}_{;\nu}$$

$$\boxed{0 = v (N u^\nu)_{;\nu}}$$

"Geometrized law of particle conservation"

The fact that the number # of particles in a volume v is constant is restated alternatively by

$$v = \frac{\#}{N}.$$

Consequently,

$$\frac{1}{N} = \frac{(\text{mean volume})}{(\text{particle})}.$$

This is the mean volume for a single particle of a fluid. Thus the 1st law of thermodynamics, Eq. (17.6) on page 17.3 is

$$p = -\frac{d(v\rho)}{dv} \quad \left(= -\frac{d(\text{energy}/\text{particle})}{d(\text{volume}/\text{particle})} \right)$$

$$= -\frac{d\left(\frac{\rho}{N}\right)}{d\left(\frac{1}{N}\right)} = N\frac{d\rho}{dN} - \rho$$

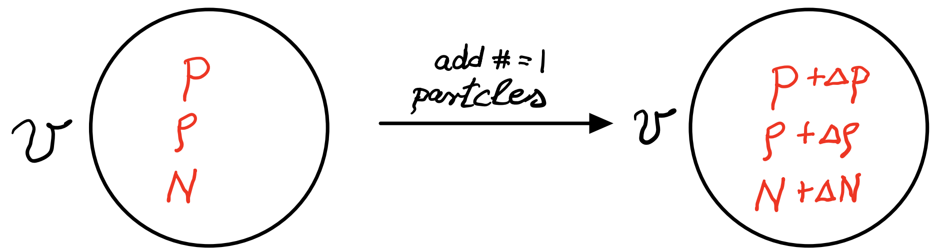
Thus,

$$\boxed{\frac{\rho}{N} + \frac{p}{N} = \frac{d\rho}{dN}}.$$

This is the "chemical potential" of a single particle, better known as the injection energy of a single particle.

To arrive at this concept from physical considerations, consider a comoving volume element with inside pressure p, energy density ρ, and particle density N.

Question: How much energy is needed to inject # = 1 particles into this volume?



Answer:

$$\frac{\Delta\rho}{\Delta N} = \frac{\Delta \frac{(\text{energy})}{(\text{volume})}\, v}{\Delta \frac{(\#\text{ of paricles})}{(\text{volume})}\, v}$$

$$= \frac{(\text{energy})}{(\text{particle})} \equiv \left(\text{"injection energy"}\right) = \left(\text{"chemical potential"}\right)$$

Let v be such that $\Delta N v = 1$. In that case $\frac{\Delta\rho}{\Delta N}$ = amount of energy necessary to inject a single particle into the fluid. This is because this energy,

$$\frac{d\rho}{dN} = \frac{p}{N} + \frac{\rho}{N},$$

consists of two parts:

$p \times \frac{1}{N}$ = work necessary to create the volume $\frac{1}{N}$ to accommodate one particle

(17,7)

$\rho \times \frac{1}{N} =$ energy that this particle must have so that once it occupies the created volume, it will be in equilibrium with the surrounding fluid.