

LECTURE 18

History of Gravitation:

Newton \rightarrow Einstein \rightarrow Cartan-Wheeler.

In MTW: Read Chapter 15

In "A Journey Into Gravity and Spacetime"

by J.A. Wheeler (available as PDF on the

internet): Read Chapter 6 & 7

Einstein's geometrodynamics is summarized by the following two statements:

1. Geometry controls the motion of matter, which is mathematized by

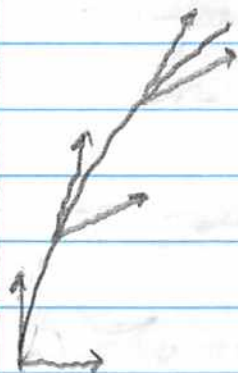
$$T^{\mu\nu}_{;\nu} = 0.$$

2. Matter controls the geometry of spacetime, which is mathematized by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu}.$$

In his line of reasoning Einstein developed a constellation of fundamental concepts that paved his way towards his geometrical formulation of gravitation.

1907:

a) Accelerated frame = 1-parameter familyof instantaneousinertial frames= 1-parameter family of
tangent spacesb) Equivalence principle according to

which a locally homogeneous, static

gravitational field is indistinguishable

from a uniformly accelerated frame.

(Hence gravitational red shift is a

Doppler shift between inertial frames

in relative motion).

(See Lecture 8 from Math 5756)

1913

(Lecture)

- a) Motion of bodies geometrized in terms of geodesics:

$$\ddot{x}^M + \Gamma^M_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0 \quad (\text{Math 5757, Lecture 4})$$

- b) Capitalize on the equivalence principle to conclude that Newton's gravitational potential ϕ is part of a spacetime tensor field:

$$g_{00} = -1 - \frac{2}{c^2} \phi \quad (\text{Math 5757, Lecture 5})$$

- c) Use momentum conservation

$$T^{mv}_{;v} = 0$$

to extend geometrized single particle motion to the motion of the most general aggregate of matter under the influence of gravitational metric potentials

d) In his 1913 paper (Lecture 5-Appendix)

Einstein recognized that geometry $g_{\mu\nu}$ including its manifestation via Newton's gravitational potential ϕ , controls the motion of matter.

But in this paper he directed attention to "the main problem of the theory of gravitation". How does one determine the geometry $g_{\mu\nu}$ if one is given the distribution and motion of matter as given by the momentum tensor T ?

How does one generalize Newton's field equations

$$\nabla^2 \phi = 4\pi G \rho$$

($G = \frac{1}{15000}$ [c.g.s. units], $\frac{1}{15000000}$ [m.k.s unit] is Newton's grav'l constant)

under the stipulation that

$$\phi \approx -\frac{1}{2} g_{00} \rightarrow \frac{1}{2} g_{\mu\nu}$$

$$\rho \rightarrow T^{\mu\nu}$$

where $T^{\mu\nu}_{; \nu} = 0$?

1916

a) (i) By means of a tour-de-force Einstein, after three years, arrived at his equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu}$$

$G_{\mu\nu} \equiv$ "Einstein tensor"

"Einstein"

where

$$G_{\mu\nu}{}^{;\nu} = 0$$

is always satisfied (it is an identity) regardless of the metric-induced curvature components that comprise $G_{\mu\nu}$.

As a consequence these Einstein field

equations always imply

$$\boxed{T^{M\nu}_{;\nu} = 0}$$

"momentum conservation"

(ii) Furthermore, relative to torsionless parallel transport, that metric-induced curvature obeys

("2nd Bianchi identity") $\cdot R^{\sigma}_{\alpha\beta\gamma;\delta} + R^{\sigma}_{\alpha\gamma\delta;\beta} + R^{\sigma}_{\alpha\delta\beta;\gamma} \equiv \frac{1}{2!} R^{\sigma}_{\alpha[\beta\gamma;\delta]} = 0$

and

("1st Bianchi identity") $R^{\sigma}_{\alpha\beta\gamma} + R^{\sigma}_{\beta\gamma\alpha} + R^{\sigma}_{\gamma\alpha\beta} \equiv R^{\sigma}_{[\alpha\beta\gamma]} = 0.$

b) (i) This system of equations

is analogous to the Maxwell field equations

$$\boxed{F^{MV};_V = 4\pi J^M} \quad \text{"Maxwell"}$$

where

$$\boxed{(F^{MV};_V)_{;M} = 0}$$

is always satisfied regardless of the details of the components that comprise the antisymmetric F^{MV} . As a consequence these Maxwell field equations always imply

$$\boxed{J^M_{;M} = 0.}$$

(ii) Furthermore, in light of Faraday's law and the non-existence of magnetic monopoles that electromagnetic tensor obeys

$$F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} \equiv \frac{1}{2!} F_{[\alpha\beta;\gamma]} = 0$$

and

$$F_{\alpha\beta} + F_{\beta\alpha} = 0$$

c) It is evident that momentum conservation and charge conservation are intrinsic features of ("hardwired" into) Einstein's geometrodynamics and Maxwell's electrodynamics.

In the face of any momentum or charge creation/annihilation the corresponding Einstein

or Maxwell's field systems would be silent: under such circumstances they do not apply.

1928 Cartan

The problem with Einstein's

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^2}T_{\mu\nu} \quad (1)$$

is that its left hand side is ill-defined when Einstein wrote it down in 1916. He constructed this tensorial equation from the components of the curvature tensor merely to (i) satisfy the momentum conservation law

$$T^{\mu\nu}_{;\mu} = 0$$

and (ii) recover the Newtonian gravitational field equation

$$\nabla^2\phi = 4\pi G\rho \quad (2)$$

in the limit of static weak gravitational fields. Such a construction is necessary but not enough.

In physics and mathematics both sides of an equation (e.g. the stress-strain relation of an elastic medium, $F=ma$, ...) must have a well-defined identity. The geometrical meaning of Einstein's l.h.s. and the line of reasoning leading to it need to be specified. Cartan's formulation in 1928, as well as Wheeler's in 1964 and in 1990 and Misner's and Wheeler's in 1972, constitute a non-trivial step forward in that direction.

A clue as to the sought-after geometrical meaning of the l.h.s. of Eq.(1) comes from the l.h.s. of Eq.(2). It expresses the following geometrical

Proposition:

For a small sphere of

$$\text{volume} = \frac{4\pi r^3}{3}$$

consider the difference between the mean value of ϕ on the surface of this sphere and its value at the center. Then

$$\nabla^2\phi = \text{moment of} \left\{ \left(\begin{array}{c} \text{mean value} \\ \text{on the boundary} \\ \text{of the sphere} \end{array} \right) - \left(\begin{array}{c} \text{value at} \\ \text{the center} \\ \text{of the sphere} \end{array} \right) \right\} \frac{8\pi}{\text{volume}}. \quad (3)$$

(Comment: This property of $\nabla^2\phi$ was pointed out by Maxwell already in 1881.) The validity of Eq.(3) is based on the following mathematical reasoning: Consider the mean value (M.V.) of the difference of the potential ϕ on the surface and the center of a small sphere of surface area $4\pi r^2$,

$$\text{M.V.} = \frac{1}{4\pi r^2} \int_0^\pi \int_0^{2\pi} [\phi(x, y, z) - \phi(0, 0, 0)] r^2 \sin\theta \, d\theta \, d\varphi \quad (4)$$

Expand $[\phi(x, y, z) - \phi(0, 0, 0)]$ to second order and obtain

$$[\dots] = x\phi_{,x} + y\phi_{,y} + z\phi_{,z} \quad (5)$$

$$+ \frac{1}{2}x^2\phi_{,xx} + \frac{1}{2}y^2\phi_{,yy} + \frac{1}{2}z^2\phi_{,zz} \quad (6)$$

$$+ xy\phi_{,xy} + yz\phi_{,yz} + zx\phi_{,zx}, \quad (7)$$

where all partial derivatives are evaluated at the center $(0, 0, 0)$ of the sphere. Introduce the spherical coordinates for (x, y, z) , do the integration and find that all linear and off-diagonal quadratic terms integrate to zero. The diagonal terms involving x^2 , y^2 , and z^2 yield identical results. One obtains

$$(\text{M.V.}) = \frac{1}{4\pi r^2} \frac{1}{2} \frac{4\pi r^4}{3} (\phi_{,xx} + \phi_{,yy} + \phi_{,zz})|_{(0,0,0)}. \quad (8)$$

Consequently, the moment of this mean deviation is

$$r \times (\text{M.V.}) = \frac{1}{8\pi} (\text{volume}) \nabla^2 \phi \Big|_{(0,0,0)} \quad (9)$$

or

$$\nabla^2 \phi = r \times (\text{M.V.}) \frac{8\pi}{(\text{volume})} \quad (10)$$

Compare this moment of mean value expression with Newton's gravitational field equation Eq.(2). One obtains for a small sphere of radius r

$$r \times (\text{M.V.}) = \frac{G}{2} \rho (\text{volume}) = \frac{G}{2} (\text{mass}) \quad (11)$$

$$r \times \{16\pi(\text{M.V.})\} = \frac{8\pi G}{c^2} (\text{mass} \cdot c^2) \quad (12)$$

Thus, Newton's gravitational field equation integrated over a sphere of volume $\frac{4\pi r^3}{3}$ is

$$\text{moment of } \left\{ 16\pi \left[\begin{array}{c} \text{deviation of} \\ \text{the surface} \\ \text{mean value of} \\ \text{the gravitational} \\ \text{potential on the} \\ \text{boundary of a} \\ \text{3-volume away} \\ \text{from its value} \\ \text{at the center} \\ \text{of that 3-volume} \end{array} \right] \right\} = \frac{8\pi G}{c^2} \left(\begin{array}{c} \text{amount of mass} \cdot c^2 \\ \text{inside that 3-volume} \end{array} \right) \quad (13)$$

This is the integral formulation of the Newtonian differential field Eq.(2). This equation relates what is in the interior of a volume to the moment of something that is measurable on its boundary.

Cartan, Misner, and Wheeler generalize this moment-based feature to 3-cubes¹ in spacetime. With them the integral formulation of the Einstein field equations get geometrized into the form

$$\text{sum of moments of } \left\{ \begin{array}{l} \text{rotation for} \\ \text{the 6 faces} \\ \text{of a small} \\ \text{3-cube} \end{array} \right\} = \frac{8\pi G}{c^2} \left(\begin{array}{l} \text{amount of} \\ \text{momenergy} \\ \text{inside this} \\ \text{3-cube} \end{array} \right) \quad (14)$$

- (a) Each 3-cube has associated with it (i) a geometrical object, its total moment of rotation and (ii) a certain vectorial amount of momenergy, which it occupies.
- (b) The field Eqs.(1) state that the momenergy in the 3-cube is the source of the moment of rotation. Both are collinear 4-vectors with Newton's relativized gravitational constant $\frac{8\pi G}{c^2}$ as the constant of proportionality.
- (c) Each vector has 4 momenergy components, and there are four 3-cube components $(\Delta x \Delta y \Delta z, \Delta t \Delta y \Delta z, \Delta t \Delta z \Delta x, \Delta t \Delta x \Delta y)$ for each volume 4-vector. Consequently, there are $4 \times 4 = 16$ equations as compared to only one equation for the Newtonian gravitation.
- (d) To summarize: these equations say that, except for the universal factor $8\pi G/c^2$, the quantity of moment of rotation equals the amount of momenergy in each of these 3-cubes.

¹more precisely, to the components $(\Delta x \Delta y \Delta z), (\Delta t \Delta y \Delta z), (\Delta t \Delta z \Delta x), (\Delta t \Delta x \Delta y)$ of a typical "volume vector"