LECTURE 2

World lines of extremal length

0. The Twin Paradox
I. Extremal length: WHY?
II. Generalization
III. The Variational Principle

LECTURE 2

IV. Parametrization Invariance
   1. Noether's Theorem Illustrated
   2. Underdetermined System

V. Torsionless metric-compatible parallel transport
VI Constant of motion

consigned to LECTURE 3

Read § 13.4 (Geodesics as world lines of extremal proper time); for a a relevant review read Section 13.3, especially the six conclusions at its end.

0. The Twin Paradox: A Reminder

2.1

The twin "paradox" is based on comparing two (biological or any other clocks in relative motion.

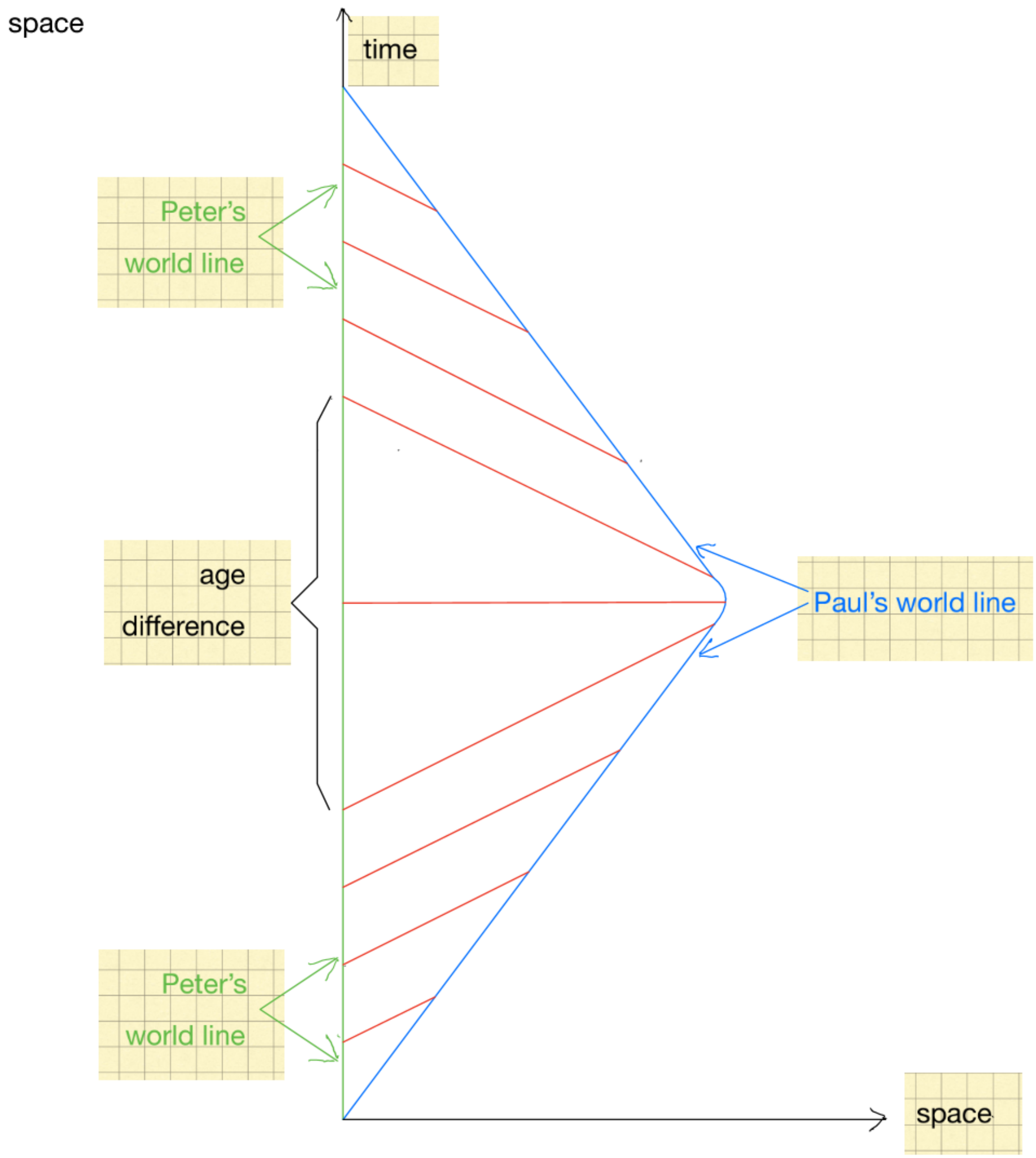

Figure 2.1: The age difference between Peter and Paul is due to the fact that Paul's lattice work of clocks records a very rapid aging in Peter during Paul's deceleration and acceleration process to be reunited with Peter. The red locus of events marks those which are simultaneous with the tickings of Paul's clock along his blue world line.

2.2

By biological standards pi-mesons, charged and neutral, have a very short life time. In their own comoving frames their life times are

$\pi^{\pm}$ life time $= 2.6 \cdot 10^{-8}$ sec

π° life time $= 8 \cdot 10^{-17}$ sec

However, these particles, when created in the upper atmosphere or by an accelerator, are usually born with very high observable velocities.

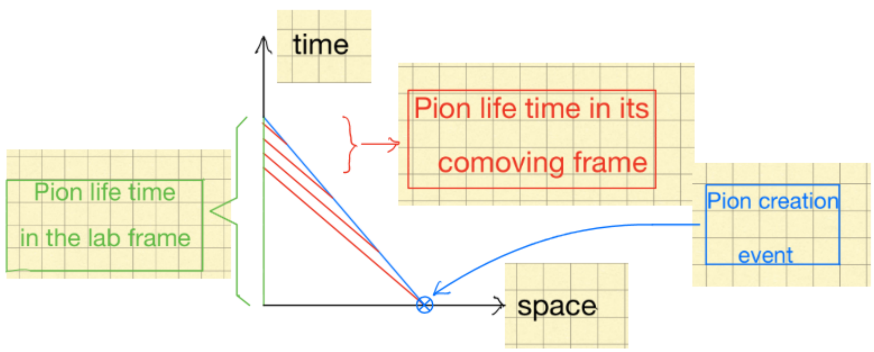

Figure 2.2 A π-meson, when created with a LAB velocity comparable to that of light, has a LAB life time considerably longer than its proper life time in its comoving frame. This time dilation relative to the LAB increases the π-meson's travel distance by a correspondingly larger amount.

Assume that the particle velocity is $v = .995c$ so that $\gamma = \frac{1}{\sqrt{1-(.995)^2}} = 10$. (2.3)

The π-meson travel distance Δx predicted within the Newtonian framework is quite different from that of its relativistic extension.

$$\pi^{\pm}: \begin{cases} \Delta X_{NEWTON} = (2.6 \times 10^{-8} sec) \times .995 \times 3 \times 10^{10} = 7.8 \times 10^{2} cm = 7.8 \text{ meter} \\ \Delta X_{RELAT.} = (2.6 \times 10^{-8} sec) \times \gamma \times .995 \times 3 \times 10^{10} = 78 \times 10^{2} cm = 78 \text{ meters} \end{cases}$$

$$\pi^{0}: \begin{cases} \Delta X_{NEWTON} = (8 \times 10^{-17} sec) \times .995 \times 3 \times 10^{10} = 2.4 \times 10^{-6} cm \\ \Delta X_{RELAT.} = (8 \times 10^{-17} sec) \times \gamma \times .995 \times 3 \times 10^{10} = 2.4 \times 10^{-5} cm \end{cases}$$

Hamilton's Principle, the Twin "Paradox", and Geodesics as Word Lines of extreme Length.

I. THE WHY OF EXTREMAL LENGTH

In a Lorentz frame it is easy to distinguish a straight line from one which is not. Compare a "broken" world line with a straight one, both starting at (0,0) and finishing at (0,T).

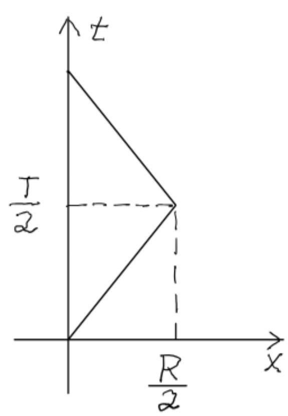

Figure 2.3 Straight vs. broken world line.

Q: What is the amount of elapsed proper time along each of these world line? (2.4)

A: Along the broken world line:

$$\tau = 2\sqrt{\left(\frac{T}{2}\right)^2 - \left(\frac{R}{2}\right)^2} = \sqrt{T^2 - R^2}$$

Along the straight world line:

$$\tau = T$$

Comment 2.1: This illustrates the twin paradox: an inertial spacetime observer ages more than one whose world line is not straight.

Conclusion: The proper time along a straight world line is a maximum in relation to time along nearby broken world lines.

II. GENERALIZATION

This conclusion generalizes to the case where one compares multiply broken world lines with a straight line in a Lorentz frame.

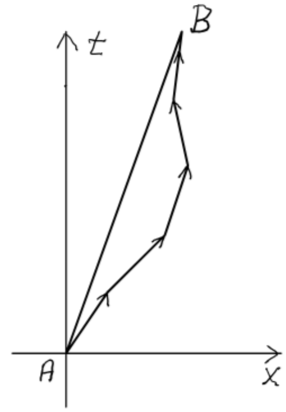

Figure 2.3 Straight vs. multiply broken world line.

In that circumstance one has

(a) $$\tau = \int_A^B d\tau = \int_A^B \sqrt{dt^2 - dx^2 - dy^2 - dz^2}$$

$$= \left(\begin{array}{l} \text{maximum for a straight line} \\ \text{compared to any variant} \\ \text{of the straight line} \end{array} \right)$$

This maximum principle holds for any Lorentz frame, even if one choses to introduce curvilinear coordinates

(b) $$\boxed{\tau_A^B = \int_A^B d\tau = \int_A^B \sqrt{-g_{\mu\nu}\, dx^\mu dx^\nu} = \left(\begin{array}{l} \text{an extremum for a time-like} \\ \text{worldline that is straight in} \\ \text{each local Lorentz frame} \\ \text{along its path, as compared to} \\ \text{any nearby variant of this} \\ \text{world line} \end{array} \right)} \qquad (2.1)$$

In a single Lorentz frame the introduction of curvilinear coordinates is optional. However, if one considers a world line passing through a sequence of distinct Lorentz frames, then the use of curvilinear coordinates is mandatory.

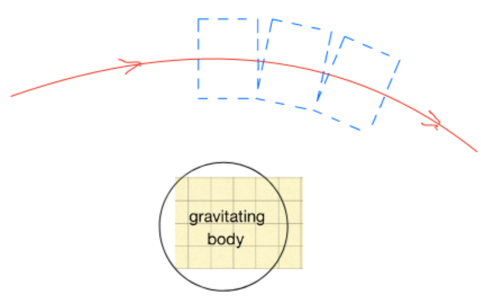

Figure 2.4 Trajectory of a particle passing through a sequence of distinct local Lorentz frames in the neighborhood of a gravitating body.

Note that in a mandatory curvilinear scenario we have replaced the "maximum" condition with an "extremum" condition. This is because there may be more than one locally straight worldline connecting the two events A and B.

III. THE VARIATIONAL PRINCIPLE

In order to determine the consequences of this variational principle, compare a locally straight world line with one of its general variants

Definition of "variant": Different in form from others of its kind.

I

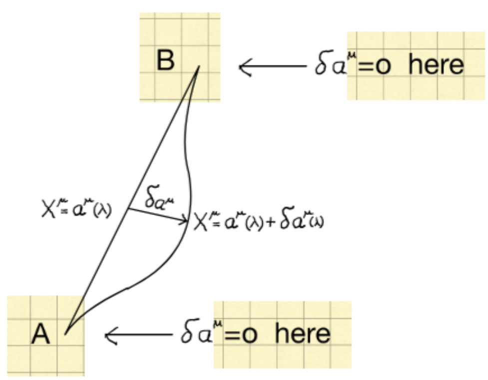

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Figure 2.5 Extremal curve $x^\mu = a^\mu(\lambda)$ and its variant $x^\mu = a^\mu(\lambda) + \delta a^\mu(\lambda)$, both passing through the same initial point event A and the final point event B.

C

Let $x^\mu = a^\mu(\lambda)$ be the world line which passes though point events A and B and **which extremizes** Eq. (2.1) on page 2.5. This implies that for any variant $x^\mu = a^\mu(\lambda) + \delta a^\mu(\lambda)$ passing through the same pair of events, A and B, the integral $\tau_A{}^B$ must satisfy

$$\delta\tau_A{}^B \equiv \tau_A{}^B[a^\mu + \delta a^\mu] - \tau_A{}^B[a^\mu] = 0 \qquad (2.2)$$

to first order accuracy in δa^μ. Here

$$\tau_A{}^B[x^\alpha] = \int_0^1 \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda \qquad (2.3)$$

for both

$$x^\mu = a^\mu(\lambda)$$

and its variant

$$x^\mu = a^\mu(\lambda) + \delta a^\mu(\lambda),$$

whose variation $\delta a^\mu(\lambda)$ vanishes at the end points A and B:

$$\delta a^\mu(0) = \delta a^\mu(1) = 0 \quad \mu = 0, 1, 2, 3, \qquad (2.4)$$

but is otherwise arbitrary.

The extremum condition Eq. (2.2) is one which is necessary for the worldline $x^\alpha = a^\alpha(\lambda)$ to be optimal. This means that it is satisfied by Eq. (2.1) on page 2.5.

In order to use Eq. (2.2), expand the integrand $I\left(a^\alpha + \delta a^\alpha; \frac{d}{d\lambda}(a^\mu + \delta a^\mu)\right) - I\left(a^\alpha; \frac{d}{d\lambda}(a^\mu)\right)$ under $\int_0^1 (\cdots)\, d\lambda$ in Eq. (2.2),

$$\Delta I \equiv \left[g_{\mu\nu}\left(a^\alpha(\lambda) + \delta a^\alpha(\lambda)\right)\frac{d(a^\mu + \delta a^\mu)}{d\lambda}\frac{d(a^\nu + \delta a^\nu)}{d\lambda}\right]^{1/2} - \left[g_{\mu\nu}\left(a^\alpha(\lambda)\right)\frac{da^\mu}{d\lambda}\frac{da^\nu}{d\lambda}\right]^{1/2},$$

in a power series in $\delta a^\alpha(\lambda)$, but retain (2.8)
only its Principal Linear Part.
The result is

$$\Delta I(\lambda) = \left[-g_{\mu\nu}(a^\alpha)\frac{da^\mu}{d\lambda}\frac{da^\nu}{d\lambda} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\delta a^\alpha \frac{da^\mu}{d\lambda}\frac{da^\nu}{d\lambda} - g_{\mu\nu}(a^\alpha)\frac{d\,\delta a^\mu}{d\lambda}\frac{da^\nu}{d\lambda} - g_{\mu\nu}(a^\alpha)\frac{da^\mu}{d\lambda}\frac{d\,\delta a^\nu}{d\lambda}\right]^{1/2} - \left[-g_{\mu\nu}(a^\alpha)\frac{da^\mu}{d\lambda}\frac{da^\nu}{d\lambda}\right]^{1/2}$$

$$= \frac{-\frac{1}{2}g_{\mu\nu}\frac{d(\delta a^\mu)}{d\lambda}\frac{da^\nu}{d\lambda} - \frac{1}{2}g_{\mu\nu}\frac{da^\mu}{d\lambda}\frac{d(\delta a^\nu)}{d\lambda} - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\delta}\delta a^\delta \frac{\partial a^\mu}{\partial\lambda}\frac{\partial a^\nu}{\partial\lambda}}{\sqrt{-g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}} \qquad (2.5)$$

Because of Eq.(2.5) the extremum condition, Eq(2.2) on page 2.7 is

$$\delta\tau_A{}^B[a^\alpha] = \int_0^1 \Delta I(\lambda)\,d\lambda = \int_0^1 \frac{-\frac{1}{2}g_{\mu\nu}\frac{d(\delta a^\mu)}{d\lambda}\frac{da^\nu}{d\lambda} - \frac{1}{2}g_{\mu\nu}\frac{da^\mu}{d\lambda}\frac{d(\delta a^\nu)}{d\lambda} - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\delta}\delta a^\delta \frac{\partial a^\mu}{\partial\lambda}\frac{\partial a^\nu}{\partial\lambda}}{\sqrt{-g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}}\,d\lambda = 0 \qquad (2.6)$$

(2.9)

The change as exhibited after the second equality sign of Eq. (2.6) is only the pricipal part linear in δa^γ of the change exhibited after the first equality sign; the contributions of quadratic and higher orders in δa^γ have been suppressed.

This is because the focus of interest is only on the necessary condition for $\tau_A{}^B[a^\alpha]$ to be an extremum, and not on what kind of extremum $\tau_A{}^B[a^\gamma]$ is.

The successful completion of this partial calculation depends on isolating δa^μ from its derivative. This is achieved by partial integration of the 1st two terms and using the fact that $\delta a^\mu = 0$ at the endpoints. The result is

$$\delta \tau_A{}^B = \frac{1}{2}\int_0^1 \left\{ \frac{d}{d\lambda}\left(\frac{g_{\mu\nu}\frac{da^\nu}{d\lambda}}{\sqrt{g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}} \right)\delta a^\mu + \frac{d}{d\lambda}\left(\frac{g_{\mu\nu}\frac{da^\mu}{d\lambda}}{\sqrt{g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}} \right)\delta a^\nu - \frac{\frac{\partial g_{\mu\nu}}{\partial x^\gamma}\frac{da^\mu}{d\lambda}\frac{da^\nu}{d\lambda}}{\sqrt{g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}}\,\delta a^\gamma \right\} d\lambda$$

$$= \int_0^1 f_\gamma(\lambda)\, \delta a^\gamma \sqrt{-g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}\, d\lambda \quad (2.7) \quad (2.10a)$$

where

$$f_\gamma(\lambda) = \frac{\frac{1}{2}}{\sqrt{-g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}}\left[\frac{d}{d\lambda}\left(\frac{g_{\gamma\nu}\frac{da^\nu}{d\lambda}}{\sqrt{g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}}\right) + \frac{d}{d\lambda}\left(\frac{g_{\gamma\nu}\frac{da^\mu}{d\lambda}}{\sqrt{g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}}\right)\right] - \frac{1}{2}\frac{\frac{\partial g_{\mu\nu}}{\partial x^\gamma}\frac{da^\mu}{d\lambda}\frac{da^\nu}{d\lambda}}{\sqrt{-g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}\sqrt{-g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}} \quad (2.8)$$

An extremum is achieved when

$$\boxed{f_\gamma(\lambda) = 0} \qquad \gamma = 0, 1, 2, 3 \qquad (2.9)$$

These equations mathematize the neccessary condition for the existence of a world line of maximal proper time running through point events A and B.

COMMENT:

At first sight Eqs (2.8) and (2.9) seem to have a daunting complexity. However, a second look reveals that they have a symmetry which not only implies an important mathematical conservation law but also reveals the key geometrical attribute of the world lines. This is because of the extremum principle on which they are based.

IV. THE EULER-LAGRANGE EQUATION

2.10b

The line of reasoning **threading** pages 2.7 to 2.10a, and culminating in Eq. (2.9) illustrates the logic at the base of the Euler-Lagrange equation

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\delta}\right)-\frac{\partial L}{\partial x^\delta}=0$$

of variational calculus. The starting point for both equations is the to-be-extremized integral

$$\tau_A{}^B=\int_{\lambda_A=0}^{\lambda_B=1}\sqrt{-g_{\mu\nu}(x^\gamma(\lambda))\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda$$

$$=\int_0^1 L(\dot{x}^\mu(\lambda),x^\gamma(\lambda))\,d\lambda$$

Its first order variation is

$$\delta\tau_A{}^B=\int_0^1\left\{L(\dot{x}^\mu(\lambda),x^\gamma(\lambda))\Big|_{x^\gamma=a^\gamma+\delta a^\gamma}-L(\dot{x}^\mu(\lambda),x^\gamma(\lambda))\Big|_{x^\gamma=a^\gamma}\right\}d\lambda \qquad (*)$$

$$=\int_0^1\left\{-\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\gamma}\right)+\frac{\partial L}{\partial x^\gamma}\right\}\Big|_{x^\gamma=a^\gamma}\Big\}\,\delta a^\gamma(\lambda)\,d\lambda. \qquad (**)$$

The expression in Eq. (**) is the Principal Linear Part (P.L.P.) of the difference as displayed by Eq. (*). It is the dominant part obtained by neglecting all non-linear terms in the series expansion of Eq. (*) in powers of δa^γ. The result is Eq. (**), namely

$$\delta\tau_A{}^B=\int_0^1 f_\gamma(\lambda)\sqrt{-g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}\,\delta a^\gamma(\lambda)\;d\lambda,$$

which is Eq. (2.7) on page 2.10a.

The fact that $\{a^\gamma(\lambda)\}$ extremizes the variational integral implies that the PLP, Eq. (**), must vanish for arbitrary variation $\delta a^\gamma(\lambda)$. Consequently,

$$\left\{-\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\gamma}\right)+\frac{\partial L}{\partial x^\gamma}\right\}\Big|_{x^\gamma=a^\gamma}=0,$$

This is the Euler-Lagrange equation, i.e.

$$f_\gamma(\lambda)\sqrt{-g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}=0$$

for the variational integral $\tau_A{}^B$.