Lecture 20

Curvature-induced Rotation on the Faces of aThree-dimensional Cube

I. Rotation

II. Curvature as a rotation applied to a 3-cube

In MTW Peruse §15.2, Fig 15.1

I. Anglar velocity-induced rotation

Rotation is the directional change enjoyed by a vectorial entity when it is moved. A rotation is quantified in terms of an angle subtended in a plane spanned by the direction of the entity before and after its change.

In three dimensions this mathematization is achieved in terms of the bivector-valued 1-form

$$\overleftrightarrow{\underset{\sim}{R}} = \frac{e_i \wedge e_j \, \epsilon^{ij}{}_k \, dx^k}{2!} \qquad (20.1)$$

It is a tensor of rank $\binom{2}{1}$, and, as such, it is a geometrical object, whose master virtue is that it is an invariant independent of one's chosen basis.

It has three unique bivectorial components

$$\frac{e_i \wedge e_j \, \epsilon^{ij}{}_1}{2!}, \quad \frac{e_i \wedge e_j \, \epsilon^{ij}{}_2}{2!}, \text{ and } \frac{e_i \wedge e_j \, \epsilon^{ij}{}_3}{2!}.$$

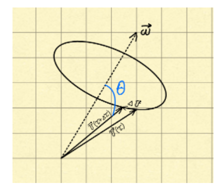
Each one, such as $e_1 \wedge e_2$, is invariant under a rotational transformation $(e_1, e_2) \rightsquigarrow (e_1', e_2')$ in its spanning plane:

$$e_1 \wedge e_2 = e_1' \wedge e_2'$$

Rotation $\underset{\sim}{\mathcal{R}}$, Eq.(20.1), has its grounding in reality by the fact that it is induced by the angular velocity $\vec{\omega}$. This means that

$$\overleftrightarrow{\underset{\sim}{\mathcal{R}}}(\Delta t\,\vec{\omega}) = \Delta t\,\frac{e_i\wedge e_j}{2!}\,\epsilon^{ij}{}_k\,\omega^k$$

produces the rotational change $\Delta\vec{v}$ when it is applied to the vector $\vec{v}$



$$\Delta\vec{v} = \overleftrightarrow{\underset{\sim}{\mathcal{R}}}(\Delta t\,\vec{\omega})\cdot\vec{v}$$

$$= \Delta t\,\frac{e_i\wedge e_j\cdot\vec{v}}{2!}\,\epsilon^{ij}{}_k\,\omega^k$$

A systematic calculation shows that the squared magnitude of $\Delta\vec{v}$ is

$$\Delta\vec{v}\cdot\Delta\vec{v} = \left(|\vec{v}|^2|\vec{\omega}|^2 - |\vec{v}\cdot\vec{\omega}|^2\right)(\Delta t)^2.$$

Consequently,

$$|\Delta\vec{v}| = |\vec{v}||\vec{\omega}|\,\sin\theta\,\Delta t.$$

This is the area of the parallelogram spanned by the vectors $\vec{v}$ and $\Delta t\,\vec{\omega}$.

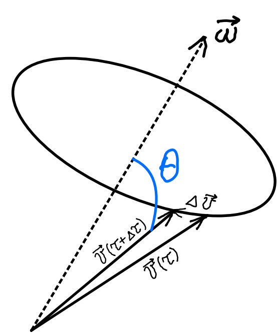


Figure 20.1 The length of $\Delta\vec{v}$ equals the area of the parallelogram spanned by $\vec{v}$ and $\Delta t\,\vec{\omega}$.

The effect of the rotation $\overset{\leftrightarrow}{\underset{\sim}{R}}(\Delta t\vec{\omega})$ on the vector $\vec{v} = v^j e_j$ is* (20.3)

$$\overset{\leftrightarrow}{R}\cdot\vec{v} = e_i\wedge e_j\,\epsilon^{ij}{}_k\,\Delta t\omega^k/2!\;\cdot\vec{v}$$
$$= (e_i v_j - e_j v_i)\epsilon^{ij}{}_k\,\Delta t\omega^k/2!$$
$$= (e_1 v_2 - e_2 v_1)\Delta t\omega^3 + (e_2 v_3 - e_3 v_2)\Delta t\omega^1 + (e_3 v_1 - e_1 v_3)\Delta t\omega^2$$
$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ \Delta t\omega_1 & \Delta t\omega_2 & \Delta t\omega_3 \end{vmatrix}$$
$$= \vec{v}\times\vec{\omega}\Delta t = \Delta\vec{v}$$

*\footnote{ Q: What is $\overset{\leftrightarrow}{\underset{\sim}{R}}(\Delta t\vec{\omega})\cdot\vec{v}$ relative to a general non-orthonormal basis $\{e_\ell\}$?

A: With
$$\vec{v} = e_\ell v^\ell$$
$$e_j\cdot e_\ell = g_{j\ell}$$
$$\epsilon^{ij}{}_k\,\omega^k = \epsilon^{ijk}\omega_k \quad \text{one has}$$

$$\overset{\leftrightarrow}{\underset{\sim}{R}}(\Delta t\vec{\omega})\cdot\vec{v} = \Delta t\,\frac{e_i\wedge e_j\cdot\vec{v}}{2!}\,\epsilon^{ij}{}_k\,\omega^k$$
$$= \Delta t\,\frac{(e_i g_{j\ell} - e_j g_{i\ell})v^\ell}{2!}\,\epsilon^{ijk}\omega_k$$
$$= \Delta t\; e_i v_j \omega_k \frac{[ijk]}{\sqrt{g}}$$
$$= \frac{\Delta t}{\sqrt{g}}\begin{vmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}$$

Let $\{e^*_i\}$ be the basis reciprocal to $\{e_j\}$, i.e.

$$e^*_i\cdot e_j = \delta_{ij}\;;\quad e^*_i = g^{ik}e_k\;;\; e_\ell = \sum_{i=1}^{3} g_{\ell i}\,e^*_i$$

Then
$$\vec{v} = e_\ell v^\ell$$
$$= \sum_i e^*_i\, g_{i\ell}\, v^\ell$$
$$= \sum_i e^*_i\, v_i$$

Thus, $\{v_i\}$ are the component of $\vec{v}$ relative to the reciprol basis $\{e_i^*\}$. These are the components v_i and w_k that go into the entries of the cross product of $\vec{v}$ and $\vec{w}$ relative to a generic non-orthogonal basis

$$\vec{v}\times\vec{w} = \underset{\sim\sim}{\overleftrightarrow{R}}(\vec{w})\cdot\vec{v} = \frac{1}{\sqrt{g}}\begin{vmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}$$

$$= e_i\, v_j\, \omega_k\, \epsilon^{ijk}$$

$$= e_i v_j \omega_k\, g^{ia} g^{jb} g^{kc}\, \epsilon_{abc}$$

$$= e_i\, g^{ia}\, v^b v^c \epsilon_{abc}$$

$$= \sqrt{g}\begin{vmatrix} e_1^* & e_2^* & e_3^* \\ v^1 & v^2 & v^3 \\ \omega^1 & \omega^2 & \omega^3 \end{vmatrix}$$

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II) CURVATURE-INDUCED ROTATION.

The concept of rotation defined by

$$\overleftrightarrow{R} = \frac{\vec{e}_i \wedge \vec{e}_j \, R^{ij}}{2!}$$

generalizes to four (and higher) dimension of spaces with an inner product (i.e. metric) structure. Indeed, recall the curvature-induced rotational change associated with a $\mathbb{u}$-$\mathbb{v}$ spanned face of a cube:

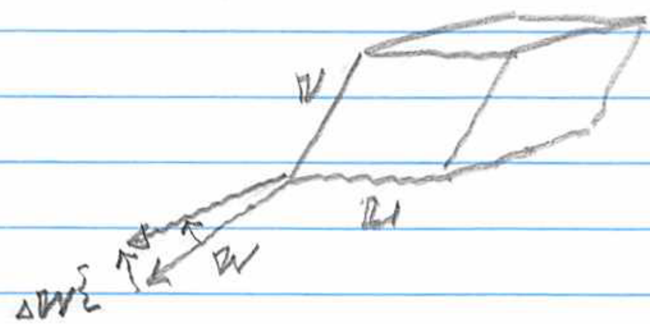

Figure 20.2

One has

$$\underset{\sim}{\overleftrightarrow{R}}(u,v):\ V \longrightarrow V$$

$$W = e_k w^k \rightsquigarrow \Delta W = e_\ell R^\ell{}_{k\alpha\beta} \frac{dx^\alpha \wedge dx^\beta(u,v)}{2!} w^k =$$

$$= e_\ell R^{\ell m}{}_{\alpha\beta}\, g_{mk} \frac{dx^\alpha \wedge dx^\beta(u,v)}{2!} w^k$$

$$= e_\ell \otimes e_m \cdot e_k w^k R^{\ell m}{}_{\alpha\beta}\, dx^\alpha \wedge dx^\beta(u,v)$$

Taking advantage of metric-induced antisymmetry $R^{\ell m}{}_{\alpha\beta} = -R^{m\ell}{}_{\alpha\beta}$, one one has

$$W \rightsquigarrow \frac{e_\ell \wedge e_m}{2!} \underbrace{R^{\ell m}{}_{\alpha\beta} \frac{dx^\alpha \wedge dx^\beta(u,v)}{2!}} \cdot W$$

$$\equiv \frac{e_\ell \wedge e_m}{2!} \overbrace{\underset{\sim}{R}^{\ell m}(u,v)} \cdot W \equiv \underset{\sim}{\overleftrightarrow{R}}(u,v) \cdot W$$

Compare this with the bivector defined rotation, Eq. (20.1) on page 20.1, one arrives at $\overleftrightarrow{R}(u,v) = \frac{e_\ell \wedge e_m}{2!} R^{\ell m}{}_{\alpha\beta} \frac{u^\alpha \wedge v^\beta}{2!}$ = "rotation"

$$(20.2)$$

It is induced by the curvature in the area subtended by the vectors u and v. For infinitesimal vectors u and v, $\overleftrightarrow{\mathcal{R}}(u,v)$'s components $\mathcal{R}^{\ell m}(u,v)$ are the angles by which a vector such as w gets rotated in the (e_ℓ, e_m)-plane.

Nota bene: In the context of spacetime, rotation refers to Euclidean rotation, Lorentz rotation or any of their combinations.

For a 3-d cube permeated by curvature, each of its faces has ~~a~~ the attribute of a rotation proportional to

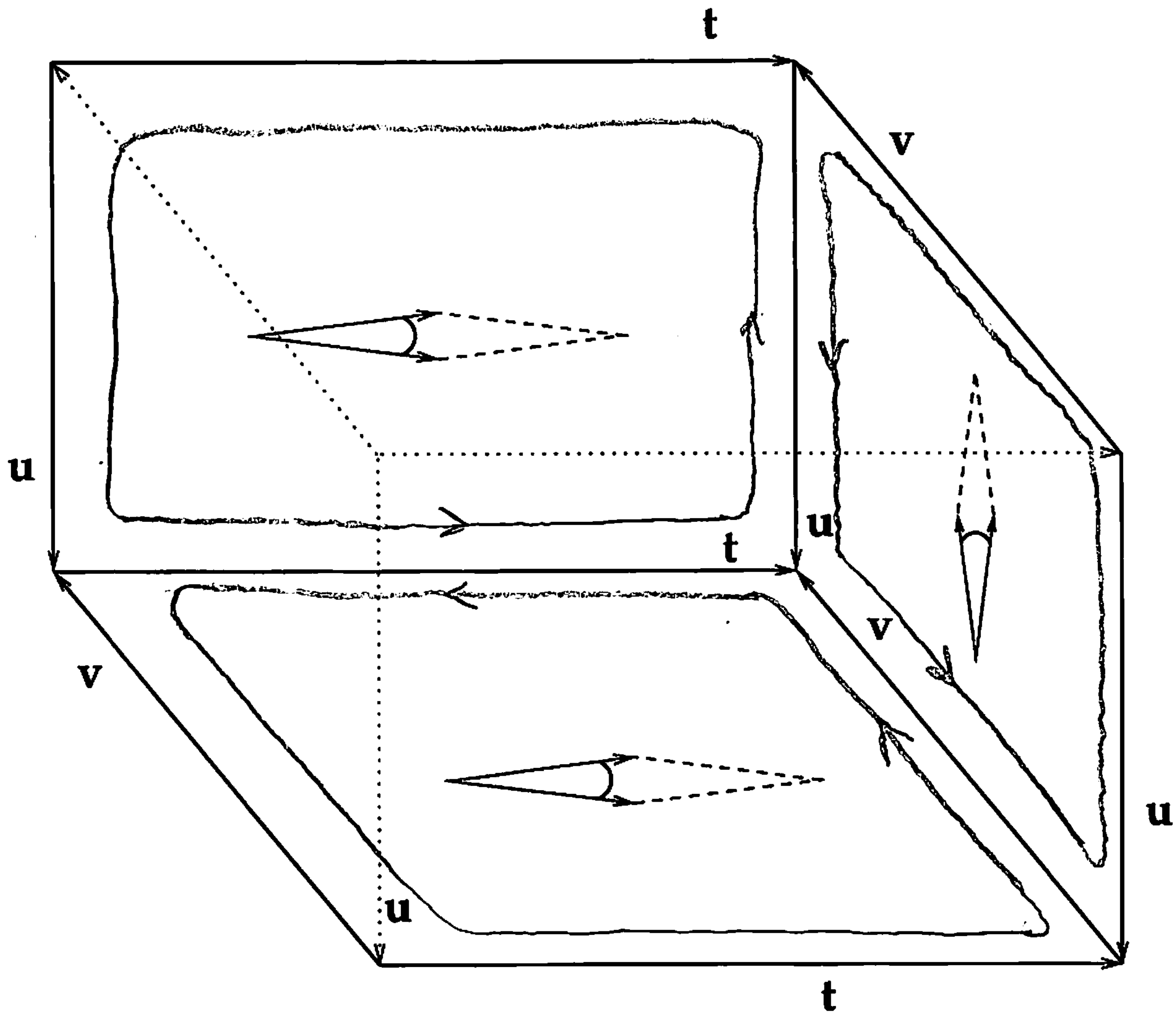

Figure 20.3 3-cube spanned by u, v, and t. Each of the cube's six faces carries a curvature-induced rotation such as Eq. (20.2) on p 20-8.

the area of the respective face.

I CURVATURE-INDUCED ROTATION APPLIED TO THE VECTOR FIELD W ON THE 6 FACES OF A 3-CUBE

The cause of gravitation is the existence of matter in any given 3-volume.

One of the conceptually most efficient ways of geometrizing gravitation is to mathematize the curvature-induced rotation on the face of a 3-d cube in 4-d spacetime.

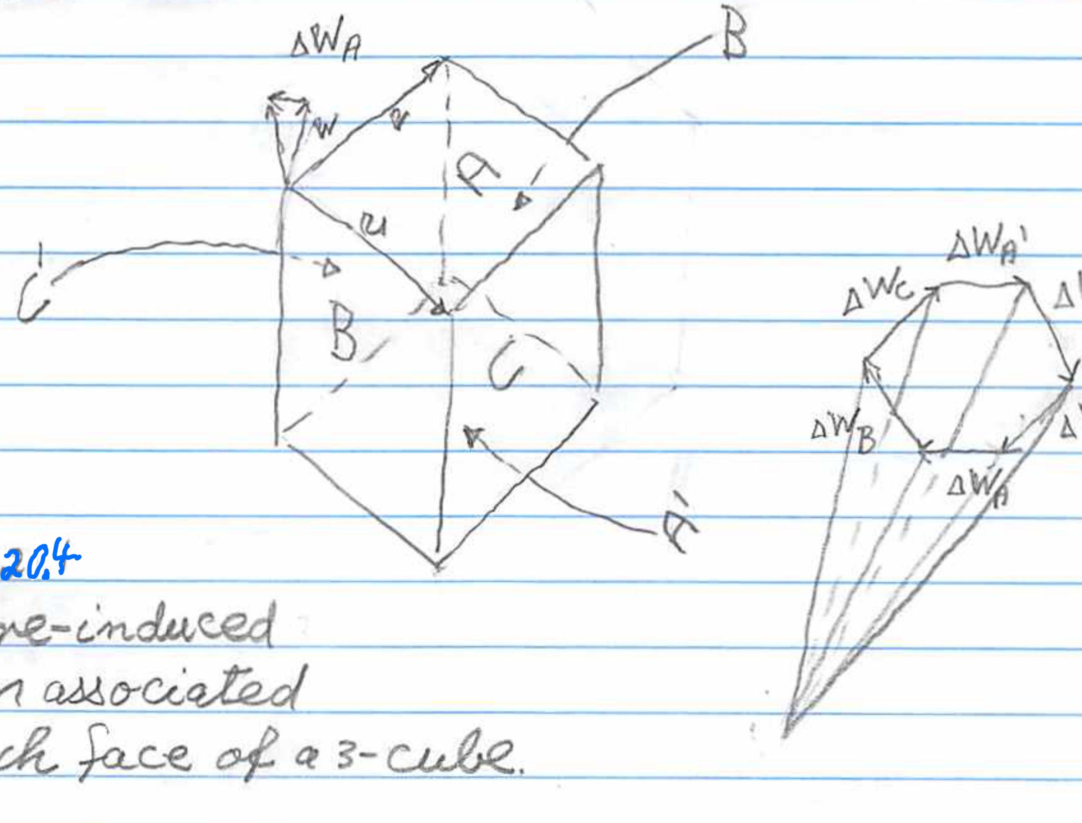

Figure 20.4
Curvature-induced rotation associated with each face of a 3-cube.

Transport the vector $\mathbf{W}$ parallel to itself around the closed square-shaped loop which bounds the face A. The result is the rotated vector $\mathbf{W}+\Delta\mathbf{W}_A$. The amount of this curvature-induced rotation is

$$\Delta\mathbf{W}_A = \frac{\mathbf{e}_\ell \wedge \mathbf{e}_m \cdot \mathbf{W}}{2!}\,\underset{\sim}{R}^{\ell m}(\mathbf{u},\mathbf{v}).$$

The sum total contribution from all six faces vanishes:

$$\Delta\mathbf{W}_A+\Delta\mathbf{W}_B+\Delta\mathbf{W}_C+\Delta\mathbf{W}_{A'}+\Delta\mathbf{W}_{B'}+\Delta\mathbf{W}_{C'}=\mathbf{0}.$$

This is because in getting parallel transported around each of the faces $\mathbf{W}$ gets moved along each edge of two abutting edges twice, but in opposite directions. The result, as depicted in Figure 20.4, is that the sum total is zero.