LECTURE 21

The Bianchi Identities

In MTW peruse §15.1 ("Bianchi Identities in Brief")

I. CURVATURE-INDUCED ROTATION FOR THE SURFACE OF A 3-CUBE

To acquire an understanding of the Einstein field equations (EFE) it is not enough to have a knowledge of curvature. One also needs an understanding of it. Rotation, in particular curvature-induced rotation, is a step into this direction.

In four dimensional spacetime consider a small 3-cube permeated by curvature.

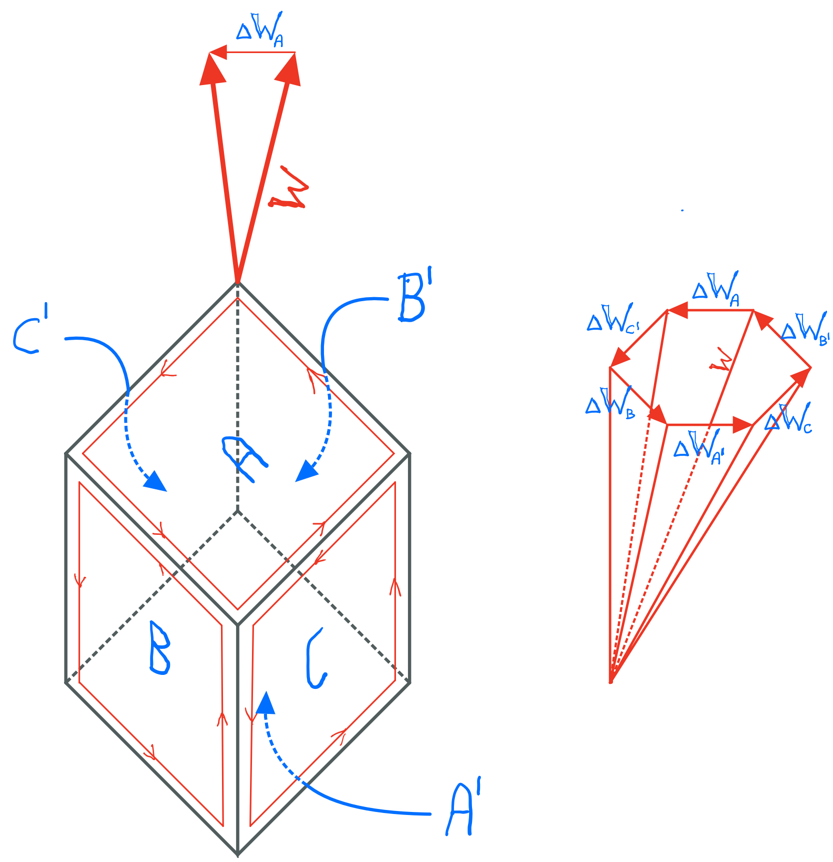

Figure 21.1 Curvature-induced rotation on each of the six faces of 3-cube.

Transport the vector $\mathbf{w}$ parallel to itself around the closed square-shaped loop which bounds the face A. The result is the rotated vector $\mathbf{w}+\Delta\mathbf{w}_A$. The amount of this curvature-induced rotation is

$$\Delta\mathbf{w}_A = \frac{\mathbf{e}_\ell \wedge \mathbf{e}_m \cdot \mathbf{w}}{2!} \; \underset{\sim}{R}^{\ell m}(\mathbf{u},\mathbf{v}).$$

The sum total contribution from all six faces vanishes:

$$\Delta\mathbf{w}_A + \Delta\mathbf{w}_B + \Delta\mathbf{w}_C + \Delta\mathbf{w}_{A'} + \Delta\mathbf{w}_{B'} + \Delta\mathbf{w}_{C'} = \mathbf{0}.$$

This is because in getting parallel transported around each of the faces $\mathbf{w}$ gets moved along each edge of two abutting edges twice, but in opposite directions. The result, as shown in Figure 21.1 on page 21.1, is that sum total is zero*:

$$\Delta W_A + \Delta W_{B'} + \Delta W_{C'} + \Delta W_{A'} + \Delta W_B + \Delta W_C = 0 \qquad (21.1)$$

* \footnote{ Note that the cause of the vanishing of this six term sum is not that terms with opposite sign such as ΔW_A and $\Delta W_{A'}$ cancel. The cause is the fact that the parallel transport of W occurs twice along each edge, but in opposite directions. That the sum is zero is due to the cancellation at each edge of abutting faces.}

To mathematize this geometrical fact consider a vector field W whose domain is on the surface of a 3-cube as well as its interior

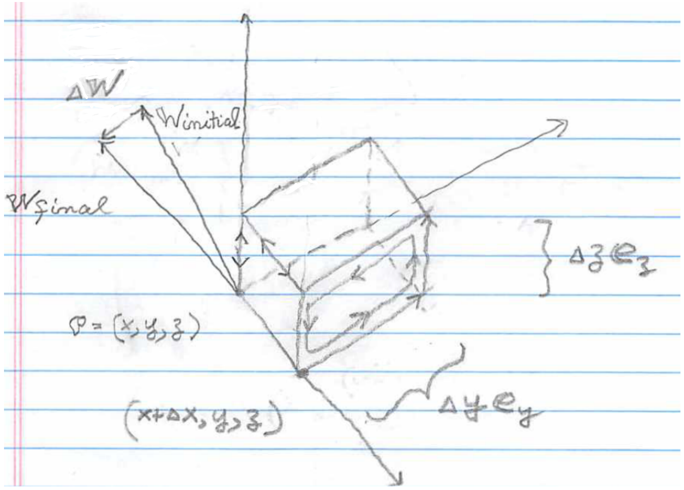

Figure 21.2 Comparing the rotations around the perimeters of two opposing faces of a 3-cube. Only one of them is depicted in the figure.

Parallel transport from a corner point along an edge to one of the 6 faces, around its boundary, and then back again to point P.

The contribution to the vectorial change from the face at $x+\Delta x$ is

$$\Delta W = e_\ell\, w^m\, R^\ell{}_{myz}(x+\Delta x, y, z)\, \Delta y\, \Delta z$$

The opposite face gives a similar contribution. The combined contribution from the pair of faces is

$$e_\ell w^m \frac{\partial}{\partial x}\left(R^\ell{}_{myz}\right) \Delta x \Delta y \Delta z \quad (\text{"front-back"})$$

Here we have used a vector basis which is parallel ($\Gamma^\mu{}_{\alpha\beta} = 0$, but $\partial_\gamma \Gamma^\mu{}_{\alpha\beta} \neq 0$ at the chosen corner P). Such a basis is induced by a "Riemann normal coordinate" systems (See Section 11.6 and Exercise 11.9 in MTW).

Relative to such a coordinate system centered at the given point, all covariant derivatives become partial derivatives at this point. This is because the basis vectors are parallel (to 2nd order accuracy) in its neighborhood.

II.) BIANCHI IDENTITIES

Other pairs of faces of that cube give similar contributions. However, the contributions from common edges of abutting faces cancel. Consequently, one has

$$0 = e_\ell \, w^m \Big(\underbrace{R^\ell{}_{myz;x}}_{\text{front-back}} + \underbrace{R^\ell{}_{mzx;y}}_{\text{right-left}} + \underbrace{R^\ell{}_{mxy;z}}_{\text{top-bottom}} \Big)$$

More generally (because of the basis independent mathematical frame work) one has

$$\boxed{0 = R^\ell{}_{mij;k} + R^\ell{}_{mjk;i} + R^\ell{}_{mki;j}} \qquad (21.2)$$

These are the "Bianchi identities".

Consider two nearby points on a curve passing through a medium permeated by a scalar f.

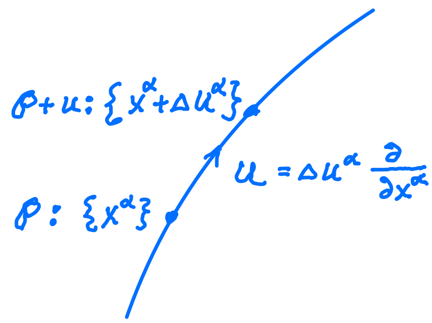

Figure 21.3 Two close-by points on a given curve.

Consider a scalar field f surrounding an infinitesimal curve segment $\overrightarrow{\mathcal{P}\,\mathcal{P}+u} \equiv u$. The difference between $f(\mathcal{P}+u)$ and $f(\mathcal{P})$ is the line integral

$$\int_{\mathcal{P}}^{\mathcal{P}+u} df = \int_{x^\alpha}^{x^\alpha+\Delta u^\alpha} \frac{\partial f}{\partial x^\beta}\, dx^\beta = \frac{\partial f}{\partial x^\alpha}\bigg|_{\mathcal{P}} \Delta u^\alpha = \left\langle \frac{\partial f}{\partial x^\alpha} dx^\alpha \,\middle|\, u \right\rangle\bigg|_{\mathcal{P}} = \nabla_u f\Big|_{\mathcal{P}}$$

Using the covariant derivative, extend this line of reasoning to vector field $\mathbf{w}$

$$\int_{\mathcal{P}}^{\mathcal{P}+u} d\mathbf{w} = \int_{x^\alpha}^{x^\alpha+\Delta u^\alpha} d\mathbf{w} = \langle d(e_\beta w^\beta) | u \rangle \equiv \nabla_u \mathbf{w}$$

Thus, the line integral of a vectorial 1-form over an infinitesimal curve segment is the covariant derivative of that vectorial one form. 21.8