Lecture 22

Curvature as rotation
permeating a contour-enclosed
area via Stokes and Cartan

In MTW read BOX 4.4: "from definite integrals to integrands"

I VECTOR-VALUED 1-FORM

(22.1)

Consider two nearby points on a curve passing through a medium permeated by a scalar f.

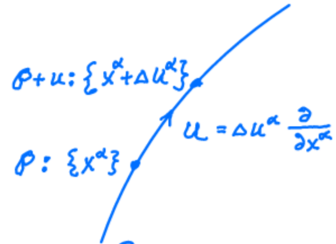

Figure 22.1 Two close-by points on a given curve.

Consider a scalar field f surrounding an infinitesimal curve segment $\overrightarrow{\mathcal{P}(\mathcal{P}+u)} = u$. The difference between $f(\mathcal{P}+u)$ and $f(\mathcal{P})$ is the scalar-valued line integral

$$\int_{\mathcal{P}}^{\mathcal{P}+u} df = \int_{x^\alpha}^{x^\alpha+\Delta u^\alpha} \frac{\partial f}{\partial x^\beta}\, dx^\beta = \frac{\partial f}{\partial x^\alpha}\Big|_{\mathcal{P}} \Delta u^\alpha = \langle \frac{\partial f}{\partial x^\alpha}\, dx^\alpha \,|\, u \rangle\Big|_{\mathcal{P}} = \nabla_u f\Big|_{\mathcal{P}}$$

Using the covariant derivative, extend this line of reasoning to vector field $\mathbf{w}$

$$\int_{\mathcal{P}}^{\mathcal{P}+u} d\mathbf{w} = \int_{x^\alpha}^{x^\alpha+\Delta u^\alpha} d\mathbf{w} = \langle d(e_\beta w^\beta) \,|\, u \rangle \equiv \nabla_u \mathbf{w}$$

Thus, the vector-valued line integral of a vectorial 1-form over an infinitesimal curve segment is the covariant derivative of that vectorial one form.

(22.2)

Consider such a line integral of the vectorial 1-form around the closed loop spanned by $u = \Delta u^\alpha \frac{\partial}{\partial x^\alpha}$ and $v = \Delta v^\beta \frac{\partial}{\partial x^\beta}$. The evaluation of this loop integral is a 5-step process.

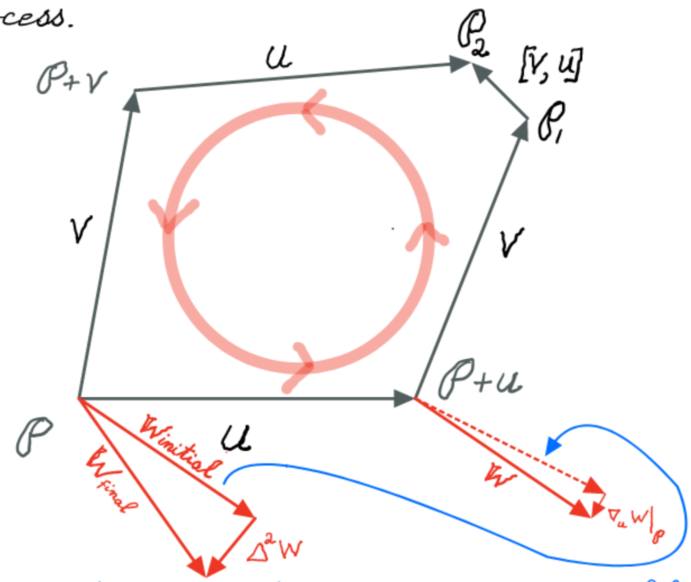

Figure 22.2 The vector-valued loop integral $\oint dW$ is a sum of line integrals

① $$\int_{\mathcal{P}}^{\mathcal{P}+u} dW + \int_{\mathcal{P}+u}^{\mathcal{P}_1} dW + \int_{\mathcal{P}_1}^{\mathcal{P}_2} dW + \int_{\mathcal{P}_2}^{\mathcal{P}+v} dW + \int_{\mathcal{P}+v}^{\mathcal{P}} dW \equiv \oint dW = \Delta^2 W$$

$$\nabla_u W\Big|_{\mathcal{P}} + \nabla_v W\Big|_{\mathcal{P}+u} + \nabla_{[v,u]} W - \nabla_u W\Big|_{\mathcal{P}+v} - \nabla_v W\Big|_{\mathcal{P}} \equiv \oint dW = \Delta^2 W$$

② Combining terms from opposite sides yields

$$\nabla_u \nabla_v W - \nabla_v \nabla_u W - \nabla_{[u,v]} W = \Delta^2 W \qquad (22.1)$$

The l.h.s. is a vector-valued line integral around the boundary of the closed polygon.

(22.3)

③ For notational economy let

$$\underset{\sim}{\vec{\Omega}} \equiv dW = d(e_\alpha w^\alpha)$$

Evaluating this vector-valued covector on the vector u, one obtains

$$\underset{\sim}{\vec{\Omega}}(v) = \langle dW | v \rangle = \nabla_v W. \qquad (22.2)$$

This is merely the extension of the directional derivative of the scalar f, $\langle df | v \rangle = \nabla_v f$, to that of a vector.

④ Introduce Eq. (22.2) into (22.1) and obtain

$$\nabla_u \underset{\sim}{\vec{\Omega}}(v) - \nabla_v \underset{\sim}{\vec{\Omega}}(u) - \underset{\sim}{\vec{\Omega}}([u,v]) = \Delta^2 W$$

Apply the 1-2 version of Stokes' theorem,*

$$\boxed{\nabla_u \underset{\sim}{\vec{\Omega}}(v) - \nabla_v \underset{\sim}{\vec{\Omega}}(u) - \underset{\sim}{\vec{\Omega}}([u,v]) = d\underset{\sim}{\vec{\Omega}}(u,v)} \qquad (22.3)$$

and obtain

$$\Delta^2 W = d\vec{\Omega}(u,v)$$

* \footnote{ Eq. (22.3) is a statement of the vectorial 1-2 Stokes' theorem in its infinitesimal frame invariant form. The scalar 1-2 Stokes' theorem has the same form: Let $\underset{\sim}{\Omega} = g\, df$ be a general scalar valued 1-form. Then the scalar-valued Stokes' theorem is

$$\nabla_u \underset{\sim}{\Omega}(v) - \nabla_v \underset{\sim}{\Omega}(u) - \underset{\sim}{\Omega}([u,v]) = d\underset{\sim}{\Omega}(u,v). \qquad (22.4) \}$$

Problem 22.1 Let $\underset{\sim}{\Omega} = g\,df$ be a scalar-valued 1-form. Show that Eq.(22.4) is valid.

Problem 22.1 Let $\vec{\underset{\sim}{\Omega}} = \xi\, df$ be a vector-valued 1-form. Show that Eq.(22.3) is valid. (22.4)

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⑤ Calculate the exterior derivative $d\vec{\underset{\sim}{\Omega}}$:

$$
\begin{aligned}
d\vec{\underset{\sim}{\Omega}} &= d\,dW = d\,d(e_\alpha w^\alpha) \\
&= d[w^\alpha de_\alpha] + d[e_\alpha dw^\alpha] \\
&= dw^\alpha \wedge de_\alpha + w^\alpha d\,de_\alpha + de_\alpha \wedge dw^\alpha + e_\alpha \underbrace{d\,d w^\alpha}_{\text{zero}} \quad (22.5) \\
&\qquad \text{cancel} \\
&= w^\alpha d\,(e_\beta \omega^\beta{}_\alpha) \\
&= w^\alpha [e_\beta\, d\omega^\beta{}_\alpha + e_\gamma \omega^\gamma{}_\beta \wedge \omega^\beta{}_\alpha]
\end{aligned}
$$

$$d\vec{\underset{\sim}{\Omega}} = e_\beta (d\omega^\beta{}_\alpha + \omega^\beta{}_\gamma \wedge \omega^\gamma{}_\alpha)\, w^\alpha$$

This is Cartan's 2nd structure equation for his curvature 2-forms

$$\underset{\sim}{\Omega}{}^\beta{}_\alpha = d\omega^\beta_\alpha + \omega^\beta_\gamma \wedge \omega^\gamma_\alpha$$

In terms of the Riemann curvature components his equation is

$$d\vec{\underset{\sim}{\Omega}} = e_\beta (R^\beta{}_{\alpha|\mu\nu|}\, dx^\mu \wedge dx^\nu)\, w^\alpha$$

Conclusion:

$$\boxed{\Delta^2 W \equiv \oint dW = d\vec{\underset{\sim}{\Omega}}(u,v) = e_\beta R^\beta{}_{\alpha\mu\nu} \Delta u^\mu \Delta v^\nu w^\alpha} \qquad (22.6)$$

Comments: (22.5)

1.) Inspite of the onslaught of derivatives on w in mathematizing $\Delta^2 w$, Eq. (22.1), the vector came through unscathed. On the contrary, the vector enjoys simply a rotation. There was no suffering from first and second order derivatives. They all cancelled out with Eq. (22.5) in the 5^{th} step.

2.) Moreover, in the presence of a metric the motion is attributed to the vector by means of a bivector, the rotation. Indeed, the change, Eq. (22.6), enjoyed by w is

$$\Delta^2 W = e_\beta R^\beta{}_{\alpha\mu\nu}\,\Delta u^\mu \Delta v^\nu\, w^\alpha$$

$$= e_\beta\, g_{\gamma\alpha} R^{\beta\gamma}{}_{|\mu\nu|}\, dx^\mu \wedge dx^\nu (u,v)\, w^\alpha.$$

Because of the metric-induced antisymmetry $R^{\beta\gamma}{}_{\mu\nu} = -R^{\gamma\beta}{}_{\mu\nu}$, $\Delta^2 W$ separates into two factors:

$$\Delta^2 W = \underbrace{e_\beta \wedge e_\gamma R^{|\beta\gamma|}{}_{|\mu\nu|}\, dx^\mu \wedge dx^\nu (u,v)}_{\overleftrightarrow{\underset{\sim}{R}}(u,v)} \cdot \underbrace{e_\alpha w^\alpha}_{W}.$$

22.6

This separation highlights the bivectorial rotation $\underset{\sim}{\overleftrightarrow{\mathcal{R}}}(u,v)$, which acts on the vector w:

$$\boxed{\Delta^2 w \equiv \oint dw = d\underset{\sim}{\vec{\Omega}}(u,v) = \underset{\sim}{\overleftrightarrow{\mathcal{R}}}(u,v)\cdot w} \qquad (22.7)$$

3.) The bivector

$$\underset{\sim}{\overleftrightarrow{\mathcal{R}}}(u,v) = \frac{e_\beta \wedge e_\alpha}{2!} R^{\beta\alpha}{}_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!}(u,v) \qquad (22.8)$$

is the rotation. It is understood to be applied to a vector w by taking the inner product:

$$\underset{\sim}{\overleftrightarrow{\mathcal{R}}}(u,v)\cdot w = \Delta^2 w.$$

This changes w by the amount $\Delta^2 w$. By leaving the vector w unspecified, this change becomes the concept

$$\underset{\sim}{\overleftrightarrow{\mathcal{R}}}(u,v) = \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu}\, \Delta u^\mu \Delta v^\nu,$$

which is Eq. (22.8), the rotation(al change) of an as-yet-unspecified vector.

4.) The rotation is a superposition of simple rotations, each taking place in the plane spanned by (22.7)

e_α and e_β. The angle of rotation in this plane is

$$R^{\alpha\beta}{}_{\mu\nu}\,\Delta u^\mu \Delta v^\nu = R^{\alpha\beta}{}_{\mu\nu}\; dx^\mu \wedge dx^\nu/2!\,(u,v).$$

The size of this angle is proportional to (i) the area spanned by the vectors u and v (ii) the magnitude of the curvature $R^{\alpha\beta}{}_{\mu\nu}$ permeating this area.

5.) By omitting explicit reference to these spanning vectors one arrives at the rank $\binom{2}{2}$ tensor

$$\overleftrightarrow{\underset{\sim}{R}} = \frac{e_\alpha \wedge e_\beta}{2!}\, R^{\alpha\beta}{}_{\mu\nu}\, \frac{dx^\mu \wedge dx^\nu}{2!},$$

the curvature-induced rotation due to an as-yet-unspecified area and of an as-yet-unspeciefied -unspecified vector.

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