Lecture 23-24

Rotation On A 3-cube
Immersed in A Curvature Field

Suggested reading: Chap. 2.8
In "Gravitation and Inertia"
by
I. Ciufolini & J.A. Wheeler

I. Rotational Change Induced by Curvature

Consider a 3-dimensional domain $\mathscr{D}$ in 4-dimensional spacetime:

$x^\alpha(u,v,t)$; $\alpha = 0,1,2,3$; $a \le u \le b$; $c \le v \le d$; $e \le t \le f$.

At each point-event $\mathscr{P} = \{x^\alpha(u,v,t)\} \in \mathscr{D}$ there are three vectors tangent to $\mathscr{D}$:

$$\mathbf{u} = \frac{\partial x^\alpha}{\partial u}\frac{\partial}{\partial x^\alpha} = u^\alpha e_\alpha$$

$$\mathbf{v} = \frac{\partial x^\beta}{\partial v}\frac{\partial}{\partial x^\beta} = v^\beta e_\beta$$

$$\mathbf{t} = \frac{\partial x^\gamma}{\partial t}\frac{\partial}{\partial x^\gamma} = t^\gamma e_\gamma$$

Also at each such point consider a 3-d infinitesimal element of volume, a 3-cube spanned by

$$\Delta u\,\mathbf{u} = \Delta u\frac{\partial x^\alpha}{\partial u}\frac{\partial}{\partial x^\alpha} = \Delta u\, u^\alpha e_\alpha$$

$$\Delta v\,\mathbf{v} = \Delta v\frac{\partial x^\beta}{\partial v}\frac{\partial}{\partial x^\beta} = \Delta v\, v^\beta e_\beta$$

$$\Delta t\,\mathbf{t} = \Delta t\frac{\partial x^\gamma}{\partial t}\frac{\partial}{\partial x^\gamma} = \Delta t\, t^\gamma e_\gamma$$

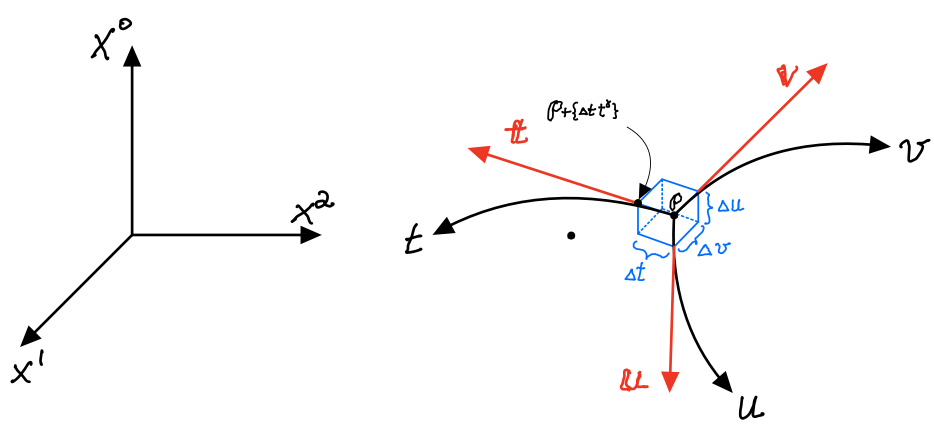


Figure 23.1 3-d cube in spacetime at point-event $\mathscr{P} = \{x^\alpha(u,v,t)\}$ where it is spanned by coordinate increments $\Delta u, \Delta v$, and Δt.

When such a 3-cube is permeated by a curvature field, each of its six faces features a rotation.

A vector $\mathbb{W}$, upon being parallel transported around the boundary of each one of these faces, will be subjected to a rotational change. Each change is proportional to the respective area of each face, to the strength of curvature field, and to the size and direction of that vector.

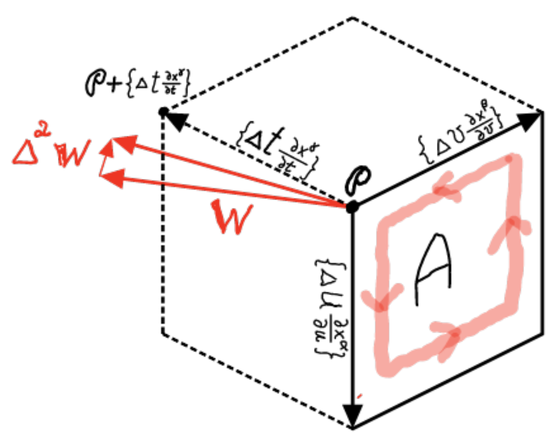


Figure 23.2 When a 3-cube is permeated by a curvature field, each of its faces features a rotation. A typical face in the figure is spanned by the two vectors,

$$\Delta u\, \mathbf{u} = \Delta u \frac{\partial x^\alpha}{\partial u}\frac{\partial}{\partial x^\alpha} \text{ and } \Delta v\, \mathbb{V} = \Delta v \frac{\partial x^\beta}{\partial v}\frac{\partial}{\partial x^\beta},$$

both belonging to the vector space at $\mathcal{P}$.

The rotation, which is featured by every face, is mathematatized by the bivector-valued 2-form

$$\overset{\leftrightarrow}{\underset{\sim}{R}}(\ ,\ ) = \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!}(\ ,\ ).$$

The rotation $\overset{\leftrightarrow}{\underset{\sim}{R}}(\Delta u\, \mathbf{u}, \Delta v\, \mathbf{v})$, when applied to the vector $\mathbf{w}$, subjects it to the rotational change

$$\Delta^2 \mathbf{w} = \overset{\leftrightarrow}{\underset{\sim}{R}}(\Delta u\, \mathbf{u}, \Delta v\, \mathbf{v}) \cdot \mathbf{w}$$
$$= e_\alpha R^\alpha{}_{\sigma\mu\nu}\, u^\mu v^\nu w^\sigma \Delta u \Delta v$$

Whichever face features a rotation, it is mathematized by the bivector-valued 2-form,

$$\overset{\leftrightarrow}{\underset{\sim}{R}} = \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!}.$$

This is a tensor of rank $\binom{2}{2}$, a multilinear map.
It resulted, we recall, from evaluating the line integral

$$\oint_{\partial A} d\mathbf{w} \equiv \oint_{\partial A} \vec{\Omega} \quad ; \quad \mathbf{w} = e_\sigma w^\sigma, \ \vec{\Omega} = d\mathbf{w}$$

around the boundary ∂A of a 2-d domain A spanned by the vectors $\Delta u\, \mathbf{u} = \Delta u \frac{\partial x^\alpha}{\partial u} \frac{\partial}{\partial x^\alpha}$ and $\Delta v\, \mathbf{v} = \Delta v \frac{\partial x^\beta}{\partial v} \frac{\partial}{\partial x^\beta}$.

Use the infinitesimal 1-2 version of Stokes' theorem

$$\nabla_{\mathbf{u}} \vec{\Omega}(\mathbf{v}) - \nabla_{\mathbf{v}} \vec{\Omega}(\mathbf{u}) - \vec{\Omega}([\mathbf{u},\mathbf{v}]) = d\vec{\Omega}(\mathbf{u},\mathbf{v}), \qquad (23.1)$$

then Cartan's 2nd structure equation to calculate $d\vec{\Omega}$.

Evaluate that 2-form on $\{\Delta u \mathbf{u}, \Delta v \mathbf{v}\}$, 23.4

$$\Delta^2 W \equiv \oint_{\partial A} dW = d\, dW (\Delta u \mathbf{u}, \Delta v \mathbf{v}), \qquad (23.2)$$

and obtain

$$= e_\alpha R^\alpha{}_{\sigma\mu\nu} u^\mu v^\nu w^\sigma \Big|_P \Delta u \Delta v \qquad (23.3)$$

The r.h.s is the Riemann sum approximation consisting of only a single term for the double integral

$$\Delta^2 W \equiv \oint_{\partial A} dW = \iint_A e_\alpha R^\alpha{}_{\sigma\mu\nu} u^\mu v^\nu w^\sigma \, du\, dv \qquad (23.4)$$

$$= \iint_A \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu} u^\mu v^\nu \cdot W \, du\, dv$$

$$\equiv \iint_A \overset{\leftrightarrow}{\underset{\sim}{R}} (u, v) \cdot W$$

over the infinitesimal 2-d domain A spanned by $(\Delta u \mathbf{u}, \Delta v \mathbf{v})$.

With these two intermediates steps, Eqs. (23.2)-(23.3), understood, one writes Eq. (23.4) as

$$\boxed{\Delta^2 W = \oint_{\partial A} dW = \iint_A d(dW)}$$

\footnote{
The infinitesimal version of the vectorial 1-2 Stokes' theorem holds for any combination of vectorial 1-forms. Without loss of generality let $\vec{\Omega} = \vec{f}\,dq$. This leads to the integral formulation of the 1-2 Stokes' theorem,

$$\int_{\partial A} \vec{\Omega} = \iint_{A} d\vec{\Omega}\,. \quad \}$$

II. Curvature-induced Rotational Change Due to the 6 Faces of a Cube

A 3-cube has 3 pairs of faces.

Q: What is the rotational change from all 3 pairs of opposing faces?

Ans.: In 3 Steps:

Step 1: From faces A' and A the rotational change is

$$\Delta^2 W_{A'} + \Delta^2 W_A =$$

$$\oint_{\partial A'} dW - \oint_{\partial A} dW = \left\{ \iint_{A'} \overleftrightarrow{\underset{\sim}{\mathcal{R}}}\Big|_{P+\Delta t\,\mathbf{t}} - \iint_{A} \overleftrightarrow{\underset{\sim}{\mathcal{R}}}\Big|_{P} \right\} \cdot W$$

$$= \left\{ \iint \overleftrightarrow{\mathcal{R}}(u,v)\Big|_{P+\Delta t\,\mathbf{t}} du\,dv - \iint \overleftrightarrow{\mathcal{R}}(u,v)\Big|_{P} du\,dv \right\} \cdot W$$

$$= \left\{ \nabla_{\mathbf{t}} \left(\overleftrightarrow{\mathcal{R}}(u,v) \right)\Big|_{P} \Delta t\,\Delta u\,\Delta v \right\} \cdot W$$

Step 2: Apply this mathematization to the other pairs of faces, B' & B and C' and C:

$$\Delta^2 W_{A'} + \Delta^2 W_A + \Delta^2 W_{B'} + \Delta^2 W_B + \Delta^2 W_{C'} + \Delta^2 W_C =$$

$$= \Big[\underbrace{\iint_{A'} - \iint_{A}} + \underbrace{\iint_{B'} - \iint_{B}} + \underbrace{\iint_{C'} - \iint_{C}} \Big] \overleftrightarrow{\mathcal{R}} \cdot W =$$

$$= \Big[\nabla_t \overleftrightarrow{\mathcal{R}}(u,v) + \nabla_u \overleftrightarrow{\mathcal{R}}(v,t) + \nabla_v \overleftrightarrow{\mathcal{R}}(t,u) \Big] \Delta t \Delta u \Delta v \} \cdot W$$

Each term is a covariant directional derivative of a sum of bivectors, tensors of rank $\binom{2}{0}$.

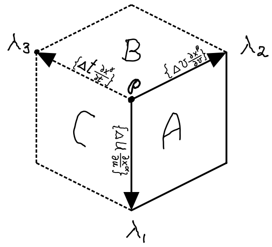
Step 3 Take advantage of the tensorial 2-3 version of Stokes' theorem

$$\nabla_t \overleftrightarrow{\mathcal{R}}(u,v) + \nabla_u \overleftrightarrow{\mathcal{R}}(v,t) + \nabla_v \overleftrightarrow{\mathcal{R}}(t,u)$$
$$-\overleftrightarrow{\mathcal{R}}([u,v],t) - \overleftrightarrow{\mathcal{R}}([v,t],u) - \overleftrightarrow{\mathcal{R}}([t,u],v) = d\overleftrightarrow{\mathcal{R}}(u,v,t)$$

and

Let $\mathcal{D} = \{ x^\alpha(\Delta u, \Delta v, \Delta t) \} \equiv \{ x^\alpha = C(\lambda_1, \lambda_2, \lambda_3) : \ 0 \leq \lambda_1, \lambda_2, \lambda_3 \leq 1 \}$

be the interior of the 3-cube



Let $\partial\mathcal{D}$ = oriented 2-d boundary of $\mathcal{D}$, the 3-cube's oriented interior spanned by t, u, and v.

$\partial\mathcal{D}$ consists of the six oriented faces of $\mathcal{D}$

$C' = C^{1+} = C(1, \lambda_2, \lambda_3)$ right (back)
$C = C^{1-} = C(0, \lambda_2, \lambda_3)$ left (front)
$B' = C^{2+} = C(\lambda_1, 1, \lambda_3)$ upper
$B = C^{2-} = C(\lambda_1, 0, \lambda_3)$ lower
$A' = C^{3+} = C(\lambda_1, \lambda_2, 1)$ left (back)
$A = C^{3-} = C(\lambda_1, \lambda_2, 0)$ right (front)

The oriented boundary $\partial\mathcal{D}$ of $\mathcal{D}$ is

$$\partial\mathcal{D} = \sum_{j=1}^{3} (-1)^{j-1}\left(C^{j+} - C^{j-}\right)$$
$$= \left(C^{1+} - C^{1-}\right) - \left(C^{2+} - C^{2-}\right) + \left(C^{3+} - C^{3-}\right)$$
$$= \left(C' - C\right) - \left(B' - B\right) + \left(A' - A\right)$$
$$\partial\mathcal{D} = \sum_{\ell=1}^{6} \left[\ell^{th}\text{ Face}\right] = \left(A' - A\right) - \left(B' - B\right) + \left(C' - C\right)$$

Appendix

Cartan's 2^{nd} structure equation

$$dd\mathbb{W} = d\vec{\Omega} = e_\alpha\left(d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta\right) w^\beta = e_\alpha R^\alpha{}_{\beta\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!} w^\beta$$

is linked to the physical world through its evaluation

Evaluate it on two spanning vectors of a face, say $\Delta u\,\mathbb{U}$ and $\Delta v\,\mathbb{V}$ depicted in Figure 23.2. Then apply it to the vector $\mathbb{W} = e_\sigma w^\sigma_\circ$,

The result is a rotational change which is enjoyed by $\mathbb{W}$. This change is

$$\Delta^2 \mathbb{W} = \overleftrightarrow{\mathcal{R}}(\Delta u\,\mathbb{U}, \Delta v\,\mathbb{V}) \cdot \mathbb{W} = \frac{\vec{e}_\alpha \wedge \vec{e}_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!}(\Delta u\,\mathbb{U}, \Delta v\,\mathbb{V}) \cdot \mathbb{W}.$$

Before evaluating it, ask and answer the following question:

Why is it rotational? The answer is this: by evaluating it, one sees that it is the sum of rotations in each of the $e_\alpha - e_\beta$ planes,

$$\underset{\sim\sim}{\overleftrightarrow{\mathcal{R}}}(\Delta u\,\mathbb{U}, \Delta v\,\mathbb{V}) \equiv \frac{e_\alpha \wedge e_\beta}{2!} \underset{\sim}{\mathcal{R}}^{\alpha\beta}(\Delta u\,\mathbb{U}, \Delta v\,\mathbb{V}) \qquad (23.1)$$

$$= \frac{e_\alpha \wedge e_\beta}{2!} \underbrace{R^{\alpha\beta}{}_{\mu\nu} u^\mu v^\nu \Delta u \Delta v}_{\theta^{\alpha\beta}} .$$

The angular amount of this rotation, which is induced by the curvature, is

$$\theta^{\alpha\beta} = R^{\alpha\beta}{}_{\mu\nu}\, u^\mu v^\nu \Delta u \Delta v.$$

This is mathematically the same as described in Lecture 19 on page 19.10, but with a physical difference. Indeed, there Eqs.(19.2) and (19.3) refer instead to rotations which are induced by the angular velocity $\vec{\omega} = e_k \omega^k$ with

$$\theta^{ij} = \epsilon^{ij}{}_k\, \omega^k \Delta t$$

as the angle of rotation in the $e_i - e_j$ plane

The evaluation of the curvature-induced change in the vector $\mathbf{w}$ is achieved by recalling that $dx^\mu(\mathbf{u}) = u^\mu$ and that $e_\beta \cdot \mathbf{w} = e_\beta \cdot e_\sigma w^\sigma = g_{\beta\sigma} w^\sigma$. The result is

$$\overset{2}{\Delta} \mathbf{w} = e_\alpha R^\alpha{}_{\sigma\mu\nu}\, u^\mu v^\nu w^\sigma \Delta u \Delta v.$$

But this is only the contribution to the vectorial change in $\mathbf{w} = e_\sigma w^\sigma$ which comes from that face of the 3-cube which is spanned by $\Delta u \mathbf{u}$ and $\Delta v \mathbf{v}$ in Figure 23.2.

Consider (i) the anti-symmetric tensor field

$$\underset{\sim}{F} = F_{\mu\nu}\, dx^\mu \wedge dx^\nu / 2!$$

permeating spacetime and (ii) a two-dimensional surface A parametrized by u and v, $x^\alpha(u, v)$.

The vectors tangent to A are

$$\mathbf{u} = \frac{\partial x^\alpha}{\partial u}\frac{\partial}{\partial x^\alpha} \equiv u^\alpha e_\alpha \;, \quad \mathbf{v} = \frac{\partial x^\beta}{\partial v}\frac{\partial}{\partial x^\beta} = v^\beta e_\beta$$

The surface integral of $\underset{\sim}{F}$ over A is mathematized by

$$\iint_A \underset{\sim}{F} \equiv \iint_A F_{\mu\nu}\, dx^\mu \wedge dx^\nu / 2! \qquad (23.3)$$

$$\equiv \iint_A \underset{\sim}{F}(\mathbf{u}, \mathbf{v})\, du\, dv$$

$$\equiv \iint_A F_{\mu\nu}(x^\sigma(u,v))\, dx^\mu \wedge dx^\nu / 2!\, (\mathbf{u}, \mathbf{v})\, du\, dv$$

$$= \iint_A F_{\mu\nu}(x^\sigma(u,v))\; u^\mu v^\nu\, du\, dv$$

When the integration domain surrounding a point $P \in A$ is so small that $F_{\mu\nu}$ is constant to lowest order, the value of the integral in the neighborhood of P is

$$\iint_A \underset{\sim}{F} = \iint_A F_{\mu\nu}(x^\sigma(u,v))\; u^\mu v^\nu\, \Delta u\, \Delta v$$

$$\boxed{\iint_A \underset{\sim}{F} \equiv \iint_A F_{\mu\nu}\, dx^\mu \wedge dx^\nu / 2! = F_{\mu\nu}\, \Delta u^\mu \Delta v^\mu \Big|_{P = \{x^\sigma(u,v)\}}} \qquad (23.4)$$