Lecture 23

Rotation On A 3-cube
Immersed In A Curvature Field

I. Rotational Change as an Area Integral (23.1)

The imprint that curvature leaves on each of the six faces of a 3-cube is a rotation generator.

For a face spanned by vectors $u = e_\alpha \Delta u^\alpha$ and $v = e_\beta \Delta u^\beta$ this generator is the bivector

$$\overleftrightarrow{\underset{\sim}{R}}(u,v) = \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!}(u,v)$$

$$= \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu} \Delta u^\mu \Delta v^\nu \qquad (23.1)$$

Its application to the vector $w = e_\gamma w^\gamma$ subjects w to the rotational change

$$\Delta^2 w = \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu} \Delta u^\mu \Delta v^\nu \cdot w \qquad (23.2)$$

$$= e_\alpha R^\alpha{}_{\beta\mu\nu} \Delta u^\mu \Delta v^\nu w^\beta$$

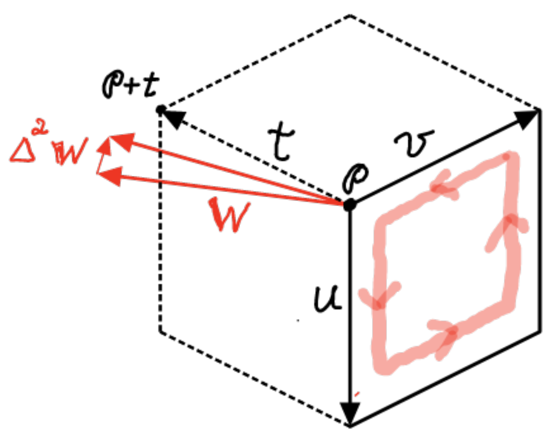

Figure 23.1 The rotation generator $\overleftrightarrow{\underset{\sim}{R}}(u,v)$ when applied to the vector w subjects it to the rotational change $\Delta^2 w$.

Consider (i) the anti-symmetric tensor field

$$\underset{\sim}{F} = F_{\mu\nu}\, dx^\mu \wedge dx^\nu / 2!$$

permeating spacetime and (ii) a two-dimensional surface A parametrized by u and v, $x^\alpha(u, v)$.

The vectors tangent to A are

$$\mathbf{u} = \frac{\partial x^\alpha}{\partial u} \frac{\partial}{\partial x^\alpha} \equiv u^\alpha e_\alpha \,, \quad \mathbf{v} = \frac{\partial x^\beta}{\partial v} \frac{\partial}{\partial x^\beta} = v^\beta e_\beta$$

The surface integral of $\underset{\sim}{F}$ over A is mathematized by

$$\iint_A \underset{\sim}{\mathbf{F}} \equiv \iint_A F_{\mu\nu}\, dx^\mu \wedge dx^\nu / 2! \qquad (23.3)$$

$$\equiv \iint_A \underset{\sim}{\mathbf{F}}\,(\mathbf{u}, \mathbf{v})\, du\, dv$$

$$\equiv \iint_A F_{\mu\nu}\left(x^\sigma(u,v)\right) dx^\mu \wedge dx^\nu / 2!\, (\mathbf{u}, \mathbf{v})\, du\, dv$$

$$= \iint_A F_{\mu\nu}\left(x^\sigma(u,v)\right)\; u^\mu v^\nu\, du\, dv$$

When the integration domain surrounding a point $\mathcal{P} \in A$ is so small that $F_{\mu\nu}$ is constant to lowest order, the value of the integral in the neighborhood of $\mathcal{P}$ is

$$\iint_A \underset{\sim}{\mathbf{F}} = \iint_A F_{\mu\nu}\left(x^\sigma(u,v)\right)\; u^\mu v^\nu\, \Delta u\, \Delta v$$

$$\boxed{\iint_A \underset{\sim}{F} \equiv \iint_A F_{\mu\nu}\, dx^\mu \wedge dx^\nu / 2! = F_{\mu\nu}\, \Delta u^\mu \Delta v^\mu \Big|_{\mathcal{P} = \{x^\sigma(u,v)\}}} \qquad (23.4)$$

Compare Eq. (23.1) with (23.3) and conclude that quantity in Eq. (23.1) is the area integral of the rotation 2-form

$$\overleftrightarrow{\mathcal{R}} = \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!}$$

over the domain A,

$$\frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu}\, \Delta u^\mu \Delta v^\nu = \iint_A \overleftrightarrow{\mathcal{R}}$$

In particular, the rotational change $\Delta^2 W$ in Eq. (23.2) is the area integral

$$\Delta^2 W = \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu}\, \Delta u^\mu \Delta v^\nu \cdot W \Big|_{P\in A} = \iint_A \overleftrightarrow{\mathcal{R}} \cdot W \quad (23.5)$$

On the other hand, using Stokes's theorem and Cartan's 2nd structure equation one obtains (Lecture 22, Eq. (22.5))

$$\Delta^2 W = \oint_{\partial A} dW = e_\alpha R^\alpha{}_{\beta\mu\nu}\, \Delta u^\mu \Delta v^\nu w^\beta$$

$$= \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu}\, \Delta u^\mu \Delta v^\nu \cdot W \qquad (23.6)$$

Compare Eqs. (23.5) and (23.6) and find that the total line (loop) integral of the rotational change in w over the boundary of the element of area A equals the area integral of the rotation generator applied to w:

$$\boxed{\oint_{\partial A} dw = \iint_A \overleftrightarrow{\underset{\sim}{\mathcal{R}}} \cdot w \equiv \iint_A \frac{e_\alpha \wedge e_\beta}{2!} \underset{\sim}{\mathcal{R}}^{\alpha\beta} \cdot w} \quad (23.7)$$

Here ∂A refers to the boundary of A as in Figure 23.1 or 22.1.

II. Curvature-induced Rotational Change Due to the 6 Faces of a Cube

A 3-cube has 3 pairs of faces.

Q: What is the rotational change from all 3 pairs of opposing faces?

Ans.: In 3 Steps:

Step 1: From faces A' and A the rotational change is

$$\oint_{\partial A'} dw - \oint_{\partial A} dw = \iint_{A'} \overleftrightarrow{\underset{\sim}{\mathcal{R}}} \cdot w \Big|_{\rho+\Delta\rho} - \iint_{A} \overleftrightarrow{\underset{\sim}{\mathcal{R}}} \cdot w \Big|_{\rho}$$

$$= \iint \overleftrightarrow{\mathcal{R}}(u,v) \Big|_{\rho+\Delta\rho} du\,dv - \iint \overleftrightarrow{\mathcal{R}}(u,v) \Big|_{\rho} du\,dv$$

$$= \nabla_t \Bigg(\underbrace{\frac{e_\alpha \wedge e_\beta}{2!} \mathcal{R}^{\alpha\beta}(u,v)}_{\overleftrightarrow{\mathcal{R}}(u,v)} \Bigg)\Bigg|_P \Delta u \, \Delta v \qquad (23.8)$$

where

$$\overleftrightarrow{\mathcal{R}}(u,v) = \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!}(u,v) = \frac{e_\alpha \wedge e_\beta}{2!} \mathcal{R}^{\alpha\beta}(u,v)$$

Step 2: *Apply this mathematization to the other pairs of faces,* B' & B *and* C' *and* C:

$$\Bigg[\underbrace{\iint_{\partial A'} - \iint_{\partial A}} + \underbrace{\iint_{\partial B'} - \iint_{\partial B}} + \underbrace{\iint_{\partial C'} - \iint_{\partial C}} \Bigg] \overleftrightarrow{\mathcal{R}} \cdot W =$$

$$= \Big[\nabla_t \overleftrightarrow{\mathcal{R}}(u,v) + \nabla_u \overleftrightarrow{\mathcal{R}}(v,t) + \nabla_v \overleftrightarrow{\mathcal{R}}(t,u) \Big] \Delta t \, \Delta u \, \Delta v \} \cdot W \qquad (23.9)$$

Each term is a covariant directional derivative of a sum of bivectors, tensors of rank $\binom{2}{0}$.

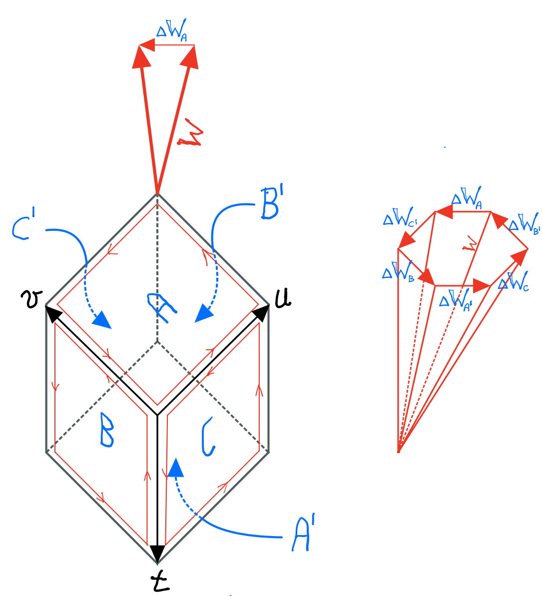

Figure 21.1 Curvature-induced rotation on each of the six faces of 3-cube.

Step 3 Take advantage of the tensorial 2-3 version of Stokes' theorem

$$\nabla_{\mathbf{t}}\, \vec{\mathcal{R}}(\mathbf{u},\mathbf{v}) + \nabla_{\mathbf{u}}\, \vec{\mathcal{R}}(\mathbf{v},\mathbf{t}) + \nabla_{\mathbf{v}}\, \vec{\mathcal{R}}(\mathbf{t},\mathbf{u}) = d\vec{\Omega}(\mathbf{u},\mathbf{v},\mathbf{t})$$

Let $\partial\mathcal{D}$ = oriented 2-d boundary of $\mathcal{D}$, the 3-cube's oriented interior spanned by $\mathbf{t}$, $\mathbf{u}$, and $\mathbf{v}$.

To be completed in class.