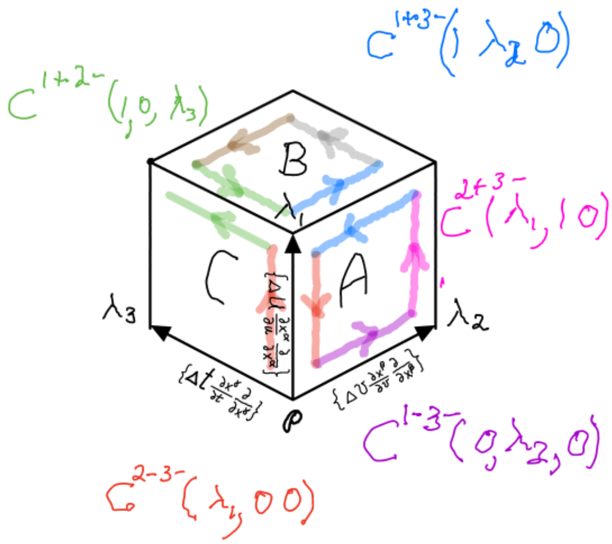
Appendix 23

①

Two equivalent restrictions on the gravitation-induced rotation field



I. The Tensorial Rotation 2-form $\overleftrightarrow{\mathcal{R}}$ is "Closed".

Consider the rotation field

$$\overleftrightarrow{\mathcal{R}} = \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!} = e_\alpha \otimes e_\beta R^{\alpha\beta}{}_{|\mu\nu|}\, dx^\mu \wedge dx^\nu$$

where

$$R^{\alpha\beta}{}_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!} = \mathcal{R}^\alpha{}_\gamma\, g^{\beta\gamma} \equiv (d\omega^\alpha{}_\gamma + \omega^\alpha{}_\sigma \wedge \omega^\sigma{}_\gamma)\, g^{\beta\gamma} \quad .$$

is the component of rotational change* within the α-β plane.

* \footnote{

Why is that change "rotational? The answer is this: By evaluating $\overleftrightarrow{\mathcal{R}}$ one obtains

$$\overleftrightarrow{\mathcal{R}}(\Delta u\, \mathbf{u}, \Delta v\, \mathbf{v}) \equiv \frac{e_\alpha \wedge e_\beta}{2!} \mathcal{R}^{\alpha\beta}(\Delta u\, \mathbf{u}, \Delta v\, \mathbf{v}) \qquad (23.1)$$

$$= \frac{e_\alpha \wedge e_\beta}{2!} \underbrace{R^{\alpha\beta}{}_{\mu\nu}\, u^\mu v^\nu \Delta u \Delta v}_{\theta^{\alpha\beta}} .$$

This is the sum of rotations in each of the e_α-e_β planes. The angular amount of this rotation, which is induced by the curvature, is (23.3)

$$\theta^{\alpha\beta} = R^{\alpha\beta}{}_{\mu\nu}\, u^\mu v^\nu \Delta u \Delta v.$$

This is mathematically the same as described in Lecture 19 on page 19.10, but with a physical difference: there Eqs. (19.2) and (19.3) refer instead to rotations which are induced by the angular velocity $\vec{\omega} = e_k \omega^k$ with

$$\theta^{ij} = \epsilon^{ij}{}_k\, \omega^k \Delta t$$

as the angle of rotation in the e_i-e_j plane.

}

Then

$$d \overset{\leftrightarrow}{\underset{\sim}{\mathcal{R}}} = d\left(e_\alpha \otimes e_\beta \underset{\sim}{\mathcal{R}}^{\alpha\beta}\right) = d\left(e_\alpha \otimes e_\beta\, g^{\beta\gamma} \mathcal{R}^\alpha{}_\gamma\right)$$

$$= d\left(e_\alpha \underset{\sim}{\mathcal{R}}^\alpha{}_\gamma\right) \otimes e_\beta\, g^{\beta\gamma} + e_\alpha \underset{\sim}{\mathcal{R}}^\alpha{}_\gamma \otimes\!\wedge\, d\left(e_\beta\, g^{\beta\gamma}\right) = \underbrace{d\left(e_\alpha \underset{\sim}{\mathcal{R}}^\alpha{}_\gamma\right)}_{①} e_\beta\, g^{\beta\gamma} + e_\alpha \underset{\sim}{\mathcal{R}}^\alpha{}_\gamma \wedge \underbrace{d\left(e_\beta\, g^{\beta\gamma}\right)}_{②} \qquad (23.A.1)$$

Notice that (i) the vectors e_α and $e_\beta g^{\beta\gamma}$ are merely (vector-valued) coefficients for the 2-form $\underset{\sim}{\mathcal{R}}^\alpha{}_\gamma$ and that (ii) the binary operations $\otimes$ and $\wedge$

commute: $\otimes \wedge = \wedge \otimes$. Because of this, e_α and $e_\beta g^{\beta\gamma}$ are mere multipliers of $\overset{③}{\mathcal{R}}{}^{\alpha}{}_{\gamma}$ and explicit reference to $\otimes$ can be dropped provided one does not interchange e_α and e_β.

The calculation of the exterior derivative ① yields the vector-valued 3-form

$$d(e_\alpha \mathcal{R}^{\alpha}{}_{\gamma}) = e_\alpha \mathcal{R}^{\alpha}{}_{\delta} \wedge \omega^{\delta}{}_{\gamma}. \qquad (23.A.2)$$

The line of reasoning leading to this depends on using the differential $de_\alpha = e_\sigma \omega^\sigma{}_\alpha$, Cartan's 2nd structure equation $\mathcal{R}^{\alpha}{}_{\delta} = d\omega^{\alpha}{}_{\delta} + \omega^{\alpha}{}_{\rho} \wedge \omega^{\rho}{}_{\delta}$, its exterior derivative, doing a cancellation, and recombining terms to obtain Eq. (23.A.1).

The calculation of the exterior derivative ② yields the vector-valued 1-form

$$d(e_\beta g^{\beta\gamma}) = -\,\omega^{\gamma}{}_{\tau}\, g^{\tau\beta} e_\beta \qquad (23.A.3)$$

which is based on taking advantage of the metric compatibility condition

$$d g_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu}$$

and of a cancellation.

Introducing Eq. (23.A.2) and (23.A.3) into (23.A.1) yields

$$d\overset{\leftrightarrow}{\mathcal{R}} \equiv d\left(e_\alpha e_\beta \mathcal{R}^{\alpha\beta}\right) = e_\alpha \mathcal{R}^{\alpha}{}_{\delta} \wedge \omega^{\delta}{}_{\gamma}\, g^{\gamma\beta} e_\beta \; - e_\alpha \mathcal{R}^{\alpha}{}_{\gamma} \wedge \omega^{\gamma}{}_{\tau}\, g^{\tau\beta} e_\beta$$

$$\boxed{d\overset{\leftrightarrow}{\mathcal{R}} = 0}$$

Thus $\overset{\leftrightarrow}{\mathcal{R}}$ is a "closed" tensorial 2-form.

Question: Does there exist a tensorial 1-form $\overset{\leftrightarrow}{A}$ such that

$$d\overset{\leftrightarrow}{A} = \overset{\leftrightarrow}{\mathcal{R}} \quad \left(= \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!}\right) \; ?$$

II. $d\overleftrightarrow{\mathcal{R}}=0 \Leftrightarrow$ Bianchi Identities

4

$\Rightarrow$: $0 = d\overleftrightarrow{\mathcal{R}} = d\left[e_\alpha \otimes e_\beta R^{\alpha\beta}{}_{\mu\nu}\, dx^\mu \wedge dx^\nu\right]$

$$= d\left[e_\alpha \left(R^{\alpha\beta}{}_{\mu\nu}\, dx^\mu \wedge dx^\nu\right) e_\beta\right]$$

STEP 1:

Recall that $de_\alpha = e_\gamma \Gamma^\gamma{}_{\alpha\sigma}\, dx^\sigma$. Thus,

$$0 = d\overleftrightarrow{\mathcal{R}} = e_\gamma \Gamma^\gamma{}_{\alpha\sigma}\, dx^\sigma \wedge (R^{\alpha\beta}{}_{\mu\nu}\, dx^\mu \wedge dx^\nu) e_\beta + e_\alpha (R^{\alpha\beta}{}_{\mu\nu,\sigma}\, dx^\sigma \wedge dx^\mu \wedge dx^\nu) e_\beta + e_\alpha (R^{\alpha\beta}{}_{\mu\nu}\, dx^\mu \wedge dx^\nu) \wedge e_\gamma \Gamma^\gamma{}_{\beta\sigma}\, dx^\sigma$$

$$= e_\alpha \left(R^{\alpha\beta}{}_{\mu\nu,\sigma} + R^{\gamma\beta}{}_{\mu\nu} \Gamma^\alpha{}_{\gamma\sigma} + R^{\alpha\gamma}{}_{\mu\nu} \Gamma^\beta{}_{\gamma\sigma}\right) e_\beta\, dx^\sigma \wedge dx^\mu \wedge dx^\nu$$

STEP 2:

Augmenting the right hand side by

$$-e_\alpha \left(R^{\alpha\beta}{}_{\delta\nu} \Gamma^\delta{}_{\mu\sigma} + R^{\alpha\beta}{}_{\mu\delta} \Gamma^\delta{}_{\nu\sigma}\right) e_\beta\, dx^\sigma \wedge dx^\mu \wedge dx^\nu \equiv 0$$

does not alter $d\overleftrightarrow{\mathcal{R}}$. This is because this augmentation is identically zero, a fact due to (i) the symmetry of the Christoffel symbol under its lower index interchange and (ii) the anti-symmetry of $R^{\alpha\beta}{}_{\mu\nu}$ under the interchange of its lower indeces.

STEP 3:

It follows that

$$0 = d\overleftrightarrow{\mathcal{R}} = e_\alpha \left(R^{\alpha\beta}{}_{\mu\nu;\sigma}\right) e_\beta\, dx^\sigma \wedge dx^\mu \wedge dx^\nu$$

Consequently,

$$\boxed{R^{\alpha\beta}{}_{\mu\nu;\sigma} + R^{\alpha\beta}{}_{\nu\sigma;\mu} + R^{\alpha\beta}{}_{\sigma\mu;\nu} = 0},$$

which are the Biamchi identies.

$\Leftarrow$: Each of the above three steps is reversible. Thus,

$$d\overleftrightarrow{\mathcal{R}} = 0 \Leftrightarrow \text{Biamchi identities indeed.}$$