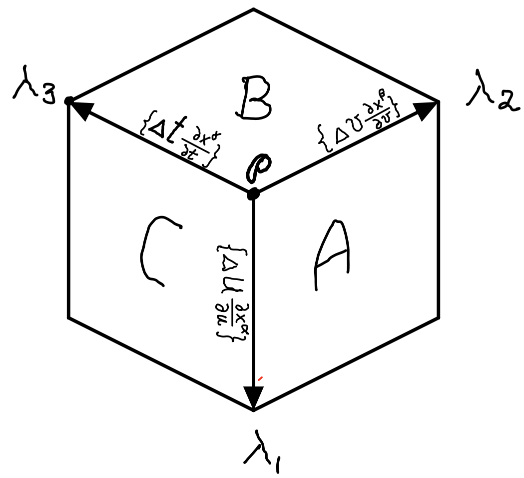
Lecture 25

Bianchi identities via

$\partial\partial = 0$

A 3-cube has 3 pairs of faces.



Q: What is the rotational change from all 3 pairs of opposing faces?

Ans.: In 3 Steps:

Step 1: From faces A' and A the rotational change is

$$\Delta^2 W_{A'} + \Delta^2 W_A =$$

$$\oint_{\partial A'} dw + \oint_{\partial A} dW = \left\{ \iint_{A'} \overleftrightarrow{\mathcal{R}}\Big|_{P+\Delta t \, t} - \iint_{A} \overleftrightarrow{\mathcal{R}}\Big|_{P} \right\} \cdot W$$

$$= \left\{ \iint \overleftrightarrow{\mathcal{R}}(u,v)\Big|_{P+\Delta t \, t} du\, dv - \iint \overleftrightarrow{\mathcal{R}}(u,v)\Big|_{P} du\, dv \right\} \cdot W$$

$$= \left\{ \nabla_t \left(\overleftrightarrow{\mathcal{R}}(u,v) \right)\Big|_{P} \Delta t \, \Delta u \, \Delta v \right\} \cdot W$$

Step 2: *Apply this mathematization to the other pairs of faces, B' & B and C' and C:*

$$\Delta^2 W_{A'} + \Delta^2 W_A + \Delta^2 W_{B'} + \Delta^2 W_B + \Delta^2 W_{C'} + \Delta^2 W_C =$$

$$\oint_{\partial A'} dW + \oint_{\partial A} dW + \oint_{\partial B'} dW + \oint_{\partial B} dW + \oint_{\partial C'} dW + \oint_{\partial C} dW = \qquad (25.1)$$

$$= \Big[\underbrace{\iint_{A'} - \iint_{A}} + \underbrace{\iint_{B'} - \iint_{B}} + \underbrace{\iint_{C'} - \iint_{C}} \Big] \overleftrightarrow{\mathcal{R}} \cdot W =$$

$$= \Big[\nabla_t \overleftrightarrow{\mathcal{R}}(u,v) + \nabla_u \overleftrightarrow{\mathcal{R}}(v,t) + \nabla_v \overleftrightarrow{\mathcal{R}}(t,u) \Big] \Delta t \Delta u \Delta v \Big\} \cdot W \qquad (25.2)$$

Each term is a covariant directional derivative of a sum of bivectors, tensors of rank $\binom{2}{0}$.

Step 3 *Take advantage of the tensorial 2-3 version of Stokes' theorem*

$$\nabla_t \overleftrightarrow{\mathcal{R}}(u,v) + \nabla_u \overleftrightarrow{\mathcal{R}}(v,t) + \nabla_v \overleftrightarrow{\mathcal{R}}(t,u)$$
$$- \overleftrightarrow{\mathcal{R}}([u,v],t) - \overleftrightarrow{\mathcal{R}}([v,t],u) - \overleftrightarrow{\mathcal{R}}([t,u],v) = \mathbf{d} \overleftrightarrow{\mathcal{R}}(u,v,t) \qquad (25.3)$$

If the spanning vector fields u, v, and t have non-zero commutators then additional line integrals need to

be added to the sum, Eq. (25.1), and hence to Eq. (25.2).
These line integrals are over paths that enclosed the areas spanned by $[\mathbf{u},\mathbf{v}]$ and $\mathbf{t}$, $[\mathbf{v},\mathbf{t}]$ and $\mathbf{u}$, as well as $[\mathbf{t},\mathbf{u}]$ and $\mathbf{v}$.
The polyhedron with these areas has more than the six faces that characterize the 3-cube on page 25.1.
Thus, for commuting spanning vectors one has

$$\Delta^2 W_{A'} + \Delta^2 W_A + \Delta^2 W_{B'} + \Delta^2 W_B + \Delta^2 W_{C'} + \Delta^2 W_C =$$

$$\oint_{\partial A'} dW + \oint_{\partial A} dW + \oint_{\partial B'} dW + \oint_{\partial B} dW + \oint_{\partial C'} dW + \oint_{\partial C} dW =$$

$$= \left[\iint_{A'} - \iint_{A} + \iint_{B'} - \iint_{B} + \iint_{C'} - \iint_{C} \right] \overleftrightarrow{\mathcal{R}} \cdot W =$$

$$= d \underset{\sim}{\overleftrightarrow{\mathcal{R}}}(\Delta u\,\mathbf{u}, \Delta v\,\mathbf{v}, \Delta t\,\mathbf{t})\Big|_P \cdot W$$

$$= \iiint_{3\text{-cube}} \underset{\sim}{d} \underset{\sim}{\overleftrightarrow{\mathcal{R}}}(\mathbf{u},\mathbf{v},\mathbf{t})\, du\, dv\, dt \cdot W$$

Together the line integrals are over $\partial\partial$(3-cube), the boundary of the boundary of the 3-cube:

$$\int_{\partial\partial(3\text{-cube})} dW = \iiint_{3\text{-cube}} \underset{\sim}{d} \underset{\sim}{\overleftrightarrow{\mathcal{R}}}(\mathbf{u},\mathbf{v},\mathbf{t})\, du\, dv\, dt \cdot W$$

However,

$$\partial\partial = 0 .$$

Thus

$$\iiint_{3\text{-cube}} d\overset{\leftrightarrow}{\mathcal{R}}(u,v,t)\, du\, dv\, dt \cdot w = 0$$

The vanishing this integral hold for arbitrarily chosen vectors u, v, t, an w. Consequently,

$$0 = d\,\underset{\sim}{\overset{\leftrightarrow}{\mathcal{R}}} \equiv d\left\{ \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!} \right\}$$

The vanishing of this exterior derivative is also validated by means of a direct calculation. It is also a way of proving the Bianchi identities.