Lecture 26

Surface force density mathematized. Translational equilibrium.

Reading assignment

1. Typeset "Lecture 24"
2. In MTW § 15.3

I. SURFACE FORCE DENSITY

Q: What is the cause of the force experienced by a cube with its plane faces?

A.: The force experienced by a 3-cube comes from the pressure and the shear stresses on each of the six faces $\Delta \vec{A}_{(\ell)}$, $\ell = 1, 2, \cdots, 6$:

$$\begin{bmatrix} \Delta F^1{}_{(\ell)} \\ \Delta F^2{}_{(\ell)} \\ \Delta F^3{}_{(\ell)} \end{bmatrix} = \begin{bmatrix} T^1{}_1 & T^1{}_2 & T^1{}_3 \\ T^2{}_1 & T^2{}_2 & T^2{}_3 \\ T^3{}_1 & T^3{}_2 & T^3{}_3 \end{bmatrix} \begin{bmatrix} \Delta A^1{}_{(\ell)} \\ \Delta A^2{}_{(\ell)} \\ \Delta A^3{}_{(\ell)} \end{bmatrix}$$

The total force from all six faces is

$$\Delta \vec{F} = \sum_{\ell=1}^{6} e_i T^i{}_j \Big|_{\ell^{th}\,\text{face}} \Delta A^j{}_{(\ell)} = \sum_{\ell=1}^{6} e_i T^i{}_j \Big|_{\ell^{th}\,\text{face}} \epsilon^j{}_{|km|} dx^k \wedge dx^m (\vec{u}_\ell, \vec{v}_\ell)$$

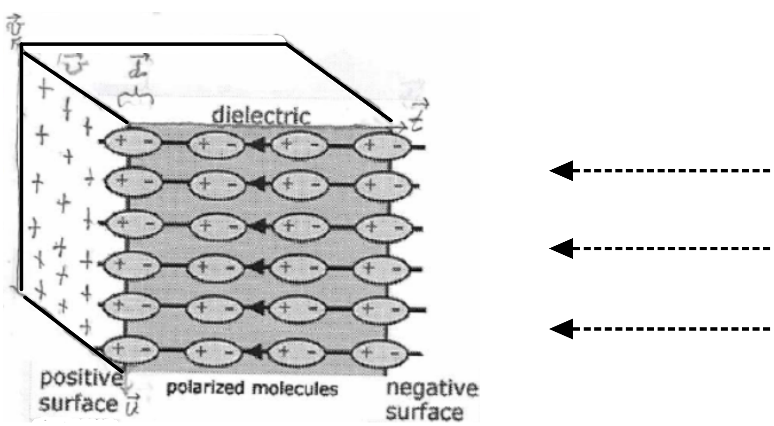
From the mechanics of a rigid body subjected to force fields one knows that they have two causal attributes:

1. Those that result in translational motion and
2. those that result in rotational motion of a

given body.*

*\footnote{The driving force behind mathematizing these two constellations of concepts is that their mathematical extension to 4-d spacetime is what is needed in order to understand the E.F.E.s in particular the l.h.s, i.e. the Einstein tensor.}

To mathematize the difference and the relation between the two, concretize the force field by an electrostatic field interacting with a dielectric.



$\vec{E} = \vec{e}_k E^k$

Figure 26.1 Polarized dielectric cube in an electric field

A dielectric consists of an array of polarizable molecules, each having a dipole moment

$$\vec p(\vec E) = e_m\, r^m(\vec E)\, q$$

when subjected to a homogeneous electrostatic field

$$\vec E = e_k E^k .$$

For a dielectric cube having volume

$$\epsilon_{|ijk|}\, dx^i\wedge dx^j\wedge dx^k(\Delta u\, \vec u, \Delta v\, \vec v, \Delta t\, \vec t\,) \simeq \begin{vmatrix} u^2 & u & u^3 \\ v^1 & v^2 & v^3 \\ t^1 & t^2 & t^3 \end{vmatrix} \Delta u\, \Delta v\, \Delta t$$

spanned by the triad $\vec u$, $\vec v$, and $\vec t$, the total number of polarized molecules is

$$N\, \epsilon_{|ijk|}\, dx^i\wedge dx^j\wedge dx^k(\Delta u\, \vec u, \Delta v\, \vec v, \Delta t\, \vec t\,),$$

where N is the measured density of molecules.

The force acting on the $(\Delta u\, \vec u, \Delta v\, \vec v)$ - spanned face* is

* \footnote{ whose normal is $\vec u \times \vec v = e_m\, \epsilon^m{}_{|ij|}\, dx^i\wedge x^j(\vec u, \vec v)$}

$$\text{"force"} = \underset{\sim}{\vec F}(\Delta u\, \vec u, \Delta v\, \vec v) = \underbrace{\vec E\, q\, \overbrace{\underbrace{N\, r^m(\vec E)\, \epsilon_{m|ij|}}_{\boxed{\text{surface density of dipoles}}}\, dx^i\wedge dx^j(\Delta u\, \vec u, \Delta v\, \vec v)}^{\boxed{\#\text{ of charges on a } (\vec u, \vec v)\text{-face}}}}_{\vec F_{|ij|}\, dx^i\wedge dx^j} \qquad (26.1)$$

The volume of a single-layered slab of molecular dipoles is

$$r^m(\vec{E})\,\epsilon_{m|ij|}\,dx^i \wedge dx^j(\vec{u},\vec{v})\,\Delta u \Delta v$$

The surface density of dipoles in this layer is a new concept. It is mathematized by the <u>surface density 2-form</u>

$$N\,r^m(\vec{E})\,\epsilon_{m|ij|}\,dx^i \wedge dx^j, \quad (\text{"surface density"})$$

and it is understood to be evaluated on a pair of vectors, in which case it yields the number dipoles in the slab.

In the presence of an electric field this dipole slab experiences the force given by Eq. (26.1).

By omitting explicit reference to that pair of vectors under the principle that Eq. (26.1) holds for <u>some</u> pair of vectors but holds for <u>any</u> pair (i.e. <u>the pair exists, but is not specified</u>), one arrives at the concept

$$\boxed{\underset{\sim}{\vec{F}} = \vec{F}_{ij}\,\frac{dx^i \wedge dx^j}{2!} = \frac{(\text{force})}{(\text{area})} = \begin{pmatrix}\text{surface force}\\ \text{density}\end{pmatrix}} \qquad (26.2)$$

Here

$$\vec{F}_{ij} = \vec{E}\,q\,N\,r^m\,\epsilon_{mij}$$

are the coordinate components of the force on an as-yet-unspecified surface area.

This rank $\binom{1}{2}$ tensor mathematizes a <u>stressfield</u>. It acts on all faces of the rigid cube. It has two causal attributes which

1. result in the cube's <u>translational</u> motion, and
2. result in the cube's <u>rotational</u> motion.

II. TRANSLATIONAL EQUILIBRIUM

The dielectric cube with zero total charge will experience a zero total force from the homogeneous stress field of a homogeneous electric field acting on the cube

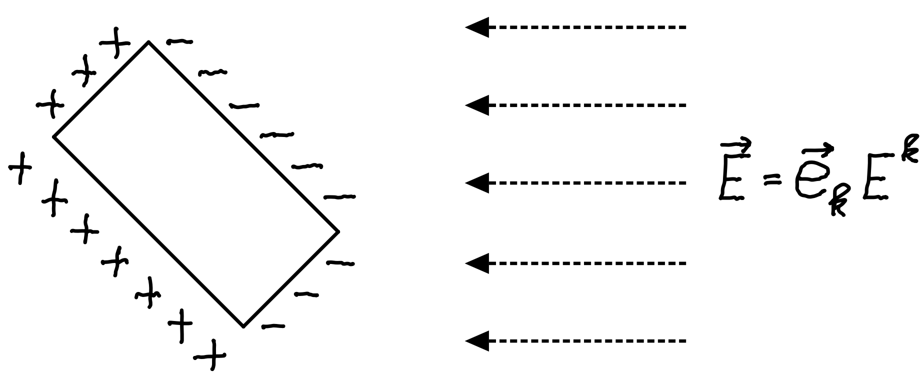


Figure 26.2 The total force due to a homogeneous electric field acting on all faces of the cube is zero.

The sum total of the forces acting on the 6 faces vanishes

$$\boxed{\vec{F}_{total} = \sum_{\ell=1}^{6} \underset{\sim}{\vec{F}}(\ell^{th}\,\text{face}) = 0} \qquad (26.3)$$

The boundary $\partial\mathcal{D}$ of the cube's interior domain consists of the union of its 6 faces,

$$\partial\mathcal{D} = \bigcup_{\ell=1}^{6} (\ell^{th}\,\text{face}),$$

and they come in pairs of opposing faces having opposite orientation. Evaluating $\underset{\sim}{\vec{F}}$ on each pair

$$(\vec{u},\vec{v}),\ (\vec{v},\vec{u});\ (\vec{v},\vec{t}),\ (\vec{t},\vec{v});\ (\vec{t},\vec{u}),\ (\vec{u},\vec{t}),$$

one finds that*

$$\sum_{\ell=1}^{6} \underset{\sim}{\vec{F}}(\ell^{th}\,\text{face}) = d\underset{\sim}{\vec{F}}(\Delta u\,\vec{u}, \Delta v\,\vec{v}, \Delta t\,\vec{t})$$

* \footnote{ The ensuing line of reasoning parallels the one leading to Eq.(25.2), page 25.2. Evaluate $\underset{\sim}{\vec{F}}$ on each of the six faces. They are located at

$\mathcal{P}+\delta\mathcal{P} = \{x^\alpha + \Delta t\, t^\alpha\}$ $\quad \mathcal{P}+\Delta\mathcal{P} = \{x^\alpha + \Delta u\, u^\alpha\}$ $\quad \mathcal{P}+d\mathcal{P} = \{x^\alpha + \Delta v\, v^\alpha\}$

and just $\mathcal{P} = \{x^\alpha\}$ for their opposing faces.

$$\sum_{\ell=1}^{6} \underset{\sim}{\vec{F}}(\ell^{th}\text{face}) = \underset{\sim}{\vec{F}}(\vec{u},\vec{v})\Big|_{\mathcal{P}+\{\Delta t\, t^\alpha\}}^{\Delta u \Delta v} - \underset{\sim}{\vec{F}}(\vec{u},\vec{v})\Big|_{\mathcal{P}}^{\Delta u \Delta v} + \underset{\sim}{\vec{F}}(\vec{v},\vec{t})\Big|_{\mathcal{P}+\{\Delta u\, u^\alpha\}}^{\Delta v \Delta t} - \underset{\sim}{\vec{F}}(\vec{v},\vec{t})\Big|_{\mathcal{P}}^{\Delta v \Delta t} + \underset{\sim}{\vec{F}}(\vec{t},\vec{u})\Big|_{\mathcal{P}+\{\Delta v\, v^\alpha\}}^{\Delta t \Delta u} - \underset{\sim}{\vec{F}}(\vec{t},\vec{u})\Big|_{\mathcal{P}}^{\Delta t \Delta u}$$

$$= \nabla_{\vec{t}}\, \underset{\sim}{\vec{F}}(\vec{u},\vec{v})\Delta t \Delta u \Delta v + \nabla_{\vec{u}}\, \underset{\sim}{\vec{F}}(\vec{v},\vec{t})\Delta u \Delta v \Delta t + \nabla_{\vec{v}}\, \underset{\sim}{\vec{F}}(\vec{t},\vec{u})\Delta v \Delta t \Delta u$$

Use the vectorial version of the 2-3 Stokes' theorem,

$$\nabla_{\vec{t}}\, \underset{\sim}{\vec{\Omega}}(\vec{u},\vec{v}) + \nabla_{\vec{u}}\, \underset{\sim}{\vec{\Omega}}(\vec{v},\vec{t}) + \nabla_{\vec{v}}\, \underset{\sim}{\vec{\Omega}}(\vec{t},\vec{u})$$

$$-\underset{\sim}{\vec{\Omega}}([\vec{u},\vec{v}],\vec{t}) - \underset{\sim}{\vec{\Omega}}([\vec{v},\vec{t}],\vec{u}) - \underset{\sim}{\vec{\Omega}}([\vec{t},\vec{u}],\vec{v}) = d\underset{\sim}{\vec{\Omega}}(\vec{u},\vec{v},\vec{t}),$$

which holds without loss of generality for

$$\underset{\sim}{\vec{\Omega}} = \vec{A}\, df \wedge dg.$$

Consequently,

$$\sum_{\ell=1}^{6} \underset{\sim}{\vec{F}}(\ell^{th}\text{face}) = d\underset{\sim}{\vec{F}}(\vec{u},\vec{v},\vec{t})\Delta u \Delta v \Delta t \quad \}$$

The condition for <u>translational equilibrium</u>, Eq.(26.3), holds for all cubes spanned by triads of vectors such as $\{u,v,t\}$. Consequently, translational equilibrium is mathematized by

$$0 = d\underset{\sim}{\vec{F}} = \left(\vec{F}_{ij;k} + \vec{F}_{jk;i} + \vec{F}_{ki;j}\right) dx^i \wedge dx^j \wedge dx^k \qquad (26.4)$$

Comment 26.1

It is an instructive exercise to show that Eq.(26.4) is equivalent to

$$0 = \vec{F}_{ij;k} + \vec{F}_{jk;i} + \vec{F}_{ki;j}\,. \qquad (26.5)$$

Appendix to Lecture 10 and 26

The vectorial measure of an as-yet-to-be specified area is

$$e_\ell\, d^2\Sigma^\ell \equiv d^2\vec{\Sigma} \equiv d^2\Sigma = e_\ell\, \epsilon^\ell{}_{ij}\, \frac{dx^i\wedge dx^j}{2!};$$

We have 1.) $e_\ell\, \epsilon^\ell{}_{ij}\, dx^i\wedge dx^j/2!\,(\vec{u},\vec{v}) \equiv \vec{u}\times\vec{v}$

and 2.) $d\,(e_\ell\, d^2\Sigma^\ell) = 0$

PROOF:

$$d\left(e_\ell\, \epsilon^\ell{}_{ij}\, \frac{dx^i\wedge dx^j}{2!}\right) = d\left(e_\ell\, g^{\ell k}\sqrt{g}\,[kij]\,\frac{dx^i\wedge dx^j}{2!}\right)$$

$$= \left[de_\ell\; g^{\ell k} + e_\ell\, dg^{\ell k} + e_\ell\, g^{\ell k}\,\frac{d\sqrt{g}}{\sqrt{g}}\right]\sqrt{g}\,[kij]\wedge\frac{dx^i\wedge dx^j}{2!}$$

$$= \Big[\underset{\textcircled{1}}{e_n\Gamma^n{}_{\ell r}\,dx^r\, g^{\ell k}} + \underset{\textcircled{2}}{e_\ell(-)g^{\ell r}g^{sk}\,dg_{rs}} + \underset{\textcircled{3}}{e_\ell\, g^{\ell k}\,\frac{d\sqrt{g}}{\sqrt{g}}}\Big]\sqrt{g}\,[kij]\,\frac{dx^i\wedge dx^j}{2!}$$

Recall that

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^r}(\sqrt{g}\,u^r) = u^r{}_{;r} = u^r{}_{,r} + u^s\,\Gamma^r{}_{sr}$$

$$u^r{}_{,r} + u^r\,\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^r} = u^r{}_{,r} + u^r\,\Gamma_r{}^s{}_s \Rightarrow \boxed{\Gamma_r{}^s{}_s = \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^r} = \frac{1}{2}g^{sm}(\cancel{g_{mr,s}} + g_{ms,r} - \cancel{g_{rs,m}})}$$

Thus

$$d\,(e_\ell\, d^2\Sigma^\ell) = e_n\Gamma^n{}_{\ell r}\, g^{\ell k}\, \underbrace{dx^r\wedge\sqrt{g}\,[kij]\,\frac{dx^i\wedge dx^j}{2!}}_{\delta^r{}_k\sqrt{g}\; dx^1\wedge dx^2\wedge dx^3} + \textcircled{2} + \textcircled{3}$$

$$= \underbrace{e_n\,\Gamma^n{}_{\ell r}\, g^{\ell r}\sqrt{g}\; dx^1\wedge dx^2\wedge dx^3}_{\textcircled{1}} + \textcircled{2} + \textcircled{3} = \textcircled{1} + \textcircled{2} + \textcircled{3}$$

$$\textcircled{1} = e_n\,\tfrac{1}{2}\,g^{nm}(g_{m\ell,r} + g_{mr,\ell} - g_{r\ell,m})\, g^{\ell r}\sqrt{g}\; dx^1\wedge dx^2\wedge dx^3$$

$$= e_n\left(g^{nm}\,g_{m\ell,r}\,g^{\ell r} - \tfrac{1}{2}\,g^{nm}\,g^{\ell r}\,g_{r\ell,m}\right)\sqrt{g}\; dx^1\wedge dx^2\wedge dx^3$$

$$\textcircled{2} = e_\ell(-)g^{\ell r}g^{sk}\,dg_{rs}\,\sqrt{g}\,[kij]\;dx^i\wedge dx^j/2!$$

$$= -e_\ell\, g^{\ell r}\, g^{sk}\, g_{rs,p}\sqrt{g}\,[kij]\;dx^p\wedge dx^i\wedge dx^j/2!$$

$$= -e_\ell\, g^{\ell r}\, g^{sk}\, g_{rs,p}\sqrt{g}\;\delta^p{}_k\; dx^1\wedge dx^2\wedge dx^3$$

$$= -e_\ell\, g^{\ell r}\, g^{sp}\, g_{rs,p}\sqrt{g}\; dx^1\wedge dx^2\wedge dx^3$$

$$= -e_n\, g^{nm}\, g^{sp}\, g_{ms,p}\sqrt{g}\; dx^1\wedge dx^2\wedge dx^3$$

$$③ = e_\ell\, g^{\ell k} \frac{1}{\sqrt{g}}\, d\sqrt{g} \wedge [k\, ij]\, dx^i \wedge dx^j / 2!$$

$$= e_\ell\, g^{\ell k} \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^p} [k\, ij]\, dx^p \wedge dx^i \wedge dx^j / 2!$$

$$= e_\ell\, g^{\ell k} \frac{1}{2}\, g^{ms}\, g_{ms,p}\, \delta^p_k \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$= e_\ell\, g^{\ell k} \frac{1}{2}\, g^{ms}\, g_{ms,k} \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3 ,$$

$$= e_n\, g^{nk} \frac{1}{2}\, g^{ms}\, g_{ms,k} \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$① + ② + ③ = e_n \left(g^{nm}\, g_{m\ell,r}\, g^{\ell r} - \frac{1}{2} g^{nm}\, g^{\ell r}\, g_{r\ell,m} \right) \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$- e_n\, g^{nm}\, g^{sp}\, g_{ms,p} \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$+ e_n\, g^{nk} \frac{1}{2}\, g^{\ell r}\, g_{\ell r,k} \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$d\left(e_\ell\, d^2\Sigma^\ell\right) = 0$$

$$d\left(e_\ell\, \epsilon^\ell{}_{ij}\, dx^i \wedge dx^j\right) = 0$$