Lecture 27

Force and Torque on a Dielectric Dipole via Cartan's Calculus

I. Fulcrums, Levers, and Moments of Force

II. Translational Equilibrium and Rotational Non-equilibrium

III. Torque as Moment of Force.

I. Fulcrums, Levers, and Moments of Force: 27.1

Force vs. Torque.

Consider a neutrally charged macroscopic body but with a non-zero surface charge, for example, an electret made out of quartz or teflon. Such a body has a dipole moment.

Figure 27.1 Macroscopic dipole in an electri force field.

When subjected to a uniform electric force field, this body will experience no net force, and thus remain in translational equilibrium, but not in rotational equilibrium. This is because the force field subjects the dipole to a "non-zero moment of force", a torque.

Equilibrium or non-equilibrium, the effect of the force field on the body's translation is via the body's surface, while the body's rotation is a volume effect.

II. Cartan's Unit Tensor $d\mathcal{P}$. (27.2)

Mathematize these effect not only by expressing them in terms of the familiar system of a fulcrum and its levers which extend to the surface areas of the body, but also by doing so in terms of Cartan's unit tensor

$$e_1 dx^1 + e_2 dx^2 + e_3 dx^3 = e_i\, dx^i = \frac{\partial}{\partial x^i}\, dx^i \equiv d\mathcal{P} \in \binom{1}{1},$$

which Cartan calls it "the displacement vector."

When it comes to geometrizing the Einstein field equations, it becomes necessary to do so in terms of lever arms and their moments, which in spacetime must be done in terms of Cartan's unit tensor: Its dictionary definition would be

$$\boxed{d\mathcal{P} = \frac{\partial}{\partial x^i} \otimes dx^i = \left(\begin{array}{c}\text{change of an}\\ \text{as-yet-unspecified}\\ \text{scalar into an}\\ \text{as-yet-unspecified}\\ \text{direction.}\end{array}\right)}$$

For a specified scalar, say ψ, the change into an as-yet-unspecified direction is

$$d\mathcal{P}(\psi,\ \ ) = \frac{\partial \psi}{\partial x^i}\, dx^i = D\psi.$$

For a specified direction, say $\vec{w} = \Delta x^k \frac{\partial}{\partial x^k}$, (27.3)
the change

$$d\varphi(\psi, \vec{w}) = \langle \frac{\partial \psi}{\partial x^i} dx^i ; \Delta x^k \frac{\partial}{\partial x^k} \rangle = \frac{\partial \psi}{\partial x^i} \delta^i_k \Delta x^k = \Delta x^k \frac{\partial \psi}{\partial x^k} = D_{\vec{w}} \psi \qquad \{27.1\}$$

III. Dielectric Dipole in a Homogeneous Electric Field.

The task of mastering the mathematical method of fulcrums, their lever arms, and their moments in terms of Cartan's unit tensor is achieved most economically with the help of a charged dipole body in a homogeneous electrostatic field.

Consider a cubical dielectric dipole spanned by vectors u, v, and t. Each of the cube's six faces has a uniform surface charge. Being immersed in a uniform electric force field, the charge on each face experiences a force. In spite of the fact that for a dielectric with no net charge the sum total force on all six faces vanishes, there is a non-zero torque.

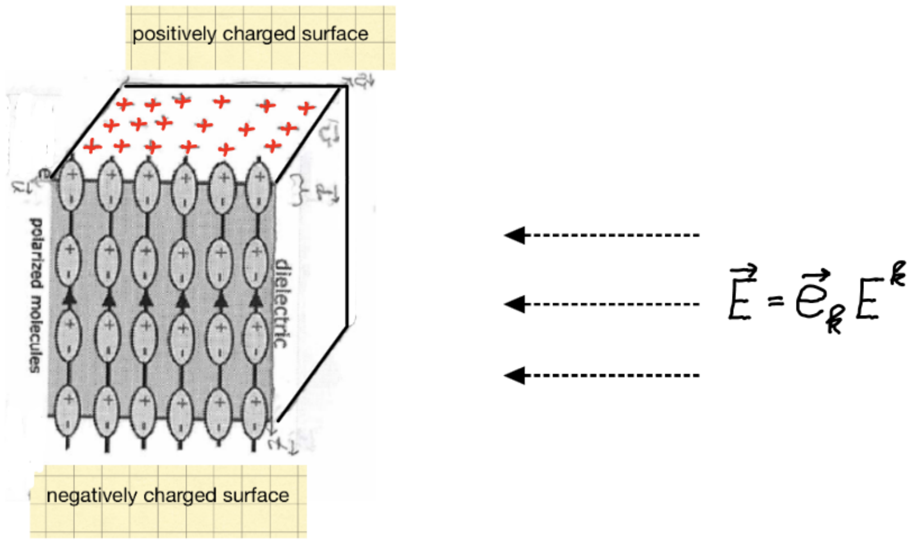


Figure 27.2 Macroscopic cubical dipole subjected to a uniform electric force field. This force force field exerts a force density, uniform but different, on each of the cube's six faces. The resulting torque is proportional to the cube's volume.

IV. Fulcrum and Lever

This torque is mathematized by an arbitrarily placed fulcrum, say P', and the lever arms emanating from it.

Let $P_3^\pm$ be two points at the center of two opposing faces of the 3-cube. The fulcrum $\wp'$ gives

27.5

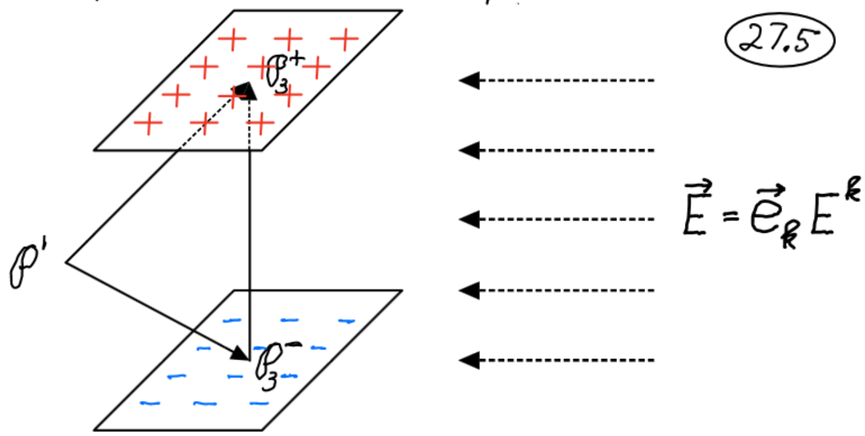


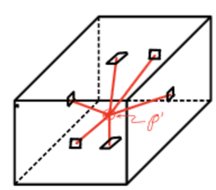
Figure 27.3 The centers between two opposing faces $\wp_3^-$ and $\wp_3^+$ are separated by the displacement vector

$$\overrightarrow{\wp_3^+ - \wp_3^-} \equiv \Delta x^3 e_3 \,.$$

The arbitrarily placed fulcrum point $\wp'$ may be inside or outside the 3-cube.

rise to 6 displacement vectors

$$\overrightarrow{\wp_i^\pm - \wp'}\,; \quad i = 1,2,3$$



between $\wp'$ and the centers of each of the six faces.

V. Moment of Force.

Next consider the two forces acting on each pair of oppositely oriented faces at the end of the two respective levers emanating from the arbitrarily placed

(27.6)

fulcrum $\mathcal{P}'$. They determine the moments of force applied to each pair of opposite faces,

$$\left(\overleftrightarrow{\mathcal{T}}\right)_3 = (\mathcal{P}_3^+ - \mathcal{P}')\wedge \underset{\sim}{\vec{F}}(\vec{u},\vec{v})\Big|_{\mathcal{P}_3^+} + (\mathcal{P}_3^- - \mathcal{P}')\wedge \underset{\sim}{\vec{F}}(\vec{v},\vec{u})\Big|_{\mathcal{P}_3^-}$$

("opposite orientation")

The vector $\underset{\sim}{\vec{F}}(\vec{u},\vec{v})\Big|_{\mathcal{P}_3^+} = \vec{F}_{|i<j|}\, dx^i \wedge x^j(\vec{u},\vec{v})\Big|_{\mathcal{P}_3^+}$ is the force acting on the face at $\mathcal{P}_3^+$.

The vector $\underset{\sim}{\vec{F}}(\vec{u},\vec{v})\Big|_{\mathcal{P}_3^-} = \vec{F}_{|i<j|}\, dx^i \wedge x^j(\vec{u},\vec{v})\Big|_{\mathcal{P}_3^-}$ is the force acting on the face at $\mathcal{P}_3^-$.

Similar expressions hold for the other four faces. The total moment of force is

$$\overleftrightarrow{\mathcal{T}} = (\mathcal{P}_3^+ - \mathcal{P}_3^-)\wedge \underset{\sim}{\vec{F}}(\vec{u},\vec{v}) + (\mathcal{P}_2^+ - \mathcal{P}_2^-)\wedge \underset{\sim}{\vec{F}}(\vec{t},\vec{u}) + (\mathcal{P}_1^+ - \mathcal{P}_1^-)\wedge \underset{\sim}{\vec{F}}(\vec{v},\vec{t}) \qquad (27.1)$$

$$-\mathcal{P}'\underbrace{\left[\underset{\sim}{\vec{F}}(\vec{u},\vec{v})\Big|_{\mathcal{P}_3^+} + \underset{\sim}{\vec{F}}(\vec{v},\vec{u})\Big|_{\mathcal{P}_3^-} + \underset{\sim}{\vec{F}}(\vec{t},\vec{u})\Big|_{\mathcal{P}_2^+} + \underset{\sim}{\vec{F}}(\vec{u},\vec{t})\Big|_{\mathcal{P}_2^-} + \underset{\sim}{\vec{F}}(\vec{v},\vec{t})\Big|_{\mathcal{P}_1^+} + \underset{\sim}{\vec{F}}(\vec{t},\vec{v})\Big|_{\mathcal{P}_1^-}\right]}_{\displaystyle \| \atop \displaystyle \sum_{\ell=1}^{6}\underset{\sim}{\vec{F}}(\ell^{th}\text{ face}) = \begin{pmatrix}\text{total force}\\ \text{on 3-cube}\end{pmatrix}}$$

When the dielectric 3-cube carries a net charge, the total force $\sum_{\ell=1}^{6}\underset{\sim}{\vec{F}}(\ell^{th}\text{ face}) \neq 0$. In that case the 3-cube is not in translational equilibrium; it will be pushed away from its initial location, and the moment of

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force $\overleftrightarrow{\mathcal{T}}$ will depend on the fulcrum $\mathcal{P}'$.

By contrast, if the dielectric carries no net charge,

$$\sum_{\ell=}^{6} \underset{\sim}{\vec{F}}(\ell^{th}\text{face}) = 0,$$

the 3-cube will be in translational equilibrium, and the moment of force, Eq. (27.1), will be independent of the location of the fulcrum $\mathcal{P}'$.

VI. Torque as a bivector-valued volume-form.

There are three pairs of opposing faces. Each pair is connected by the respective three pairs of connecting levers. As depicted in Figure 27.3, they are

$$\left.\begin{aligned} \mathcal{P}_3^+ - \mathcal{P}_3^- &= \Delta x^3\ e_3 = \vec{t} \\ \mathcal{P}_2^+ - \mathcal{P}_2^- &= \Delta x^2\ e_2 = \vec{v} \\ \mathcal{P}_1^+ - \mathcal{P}_1^- &= \Delta x^1\ e_1 = \vec{u} \end{aligned}\right\} \qquad (27.2)$$

These, together with Eq. (26.2)* [in Lecture 26], imply that the total moment of force, Eq. (27.1), is

$$\overleftrightarrow{\mathcal{T}}(\vec{u},\vec{v},\vec{t}) = \underbrace{\Delta x^3 e_3}_{\vec{t}} \wedge \vec{F}_{|ij|}\, dx^i \wedge dx^j (\underbrace{\Delta x^1 e_1}_{\vec{u}}, \underbrace{\Delta x^2 e_2}_{\vec{v}}) \qquad e_3 \langle dx^3, t\rangle \wedge \vec{F}_{|ij|}\ dx^i \wedge dx^j (u\,v)$$

$$e_2 \langle dx^2 | v\rangle \wedge \vec{F}_{|ij|}\ dx^i \wedge dx^j (t\,u)$$

$$+ \underbrace{\Delta x^2 e_2}_{\vec{v}} \wedge \vec{F}_{|ij|}\, dx^i \wedge dx^j (\underbrace{\Delta x^3 e_3}_{\vec{t}}, \underbrace{\Delta x^1 e_1}_{\vec{u}}) \qquad e_1 \langle dx^1 | u\rangle \wedge \vec{F}_{ji}\ dx^i \wedge dx^j (v, t)$$

$$+\underbrace{\Delta x^1 e_1}_{\vec u} \wedge \vec F_{|ij|}\, dx^i \wedge dx^j (\underbrace{\Delta x^2 e_2}_{\vec v}, \underbrace{\Delta x^3 e_3}_{\vec t}) \qquad (27.3) \qquad 27.8$$

\footnote{ The vectorial coefficients $\vec F_{ij}$ of the 2-form $\underset{\sim}{\vec F} = \vec F_{ij}\, dx^i \wedge dx^j/2!$ are

$$\vec F_{ij} = e_k E^k q N r^m \epsilon_{mij} \quad \}$$

In spite of superficial appearances to the contrary, the expression for the total moment of force, Eq.(27.3), is a coordinate frame invariant. Indeed, using the basis expansions Eq.(27.2), Eq.(27.3) becomes

$$\overset{\leftrightarrow}{\underset{\sim}{T}}(\vec u, \vec v, \vec t) = e_m\, dx^m \wedge \vec F_{|ij|}\, dx^i \wedge dx^j (\vec u, \vec v, \vec t),$$

which in terms of Cartan's unit tensor / "displacement vector"

$$dP = e_m\, dx^m$$

is

$$\overset{\leftrightarrow}{T}(\vec u, \vec v, \vec t) = dP \wedge \underset{\sim}{\vec F}(\vec u, \vec v, \vec t)$$

or

$$\boxed{\begin{aligned}\overset{\leftrightarrow}{\underset{\sim}{T}} &= dP \wedge \vec F_{ij}\, dx^i \wedge dx^j/2! \\ &= e_m \wedge \vec F_{ij}\, dx^m \wedge dx^i \wedge dx^j/2!\end{aligned}}$$

This moment of force is a tensor of rank $\binom{2}{3}$. Physically it is the torque exerted by the homogeneous electrostatic field $\vec E = e_k E^k$ on a 3-cube spanned by a triad of as-yet-unspecified vectors,

$$\overset{\leftrightarrow}{T} = e_m \wedge e_k E^k q N r^n \epsilon_{n|ij|}\, dx^m \wedge dx^i \wedge dx^j$$