Lecture 28

Torque Via Cartan's Moment

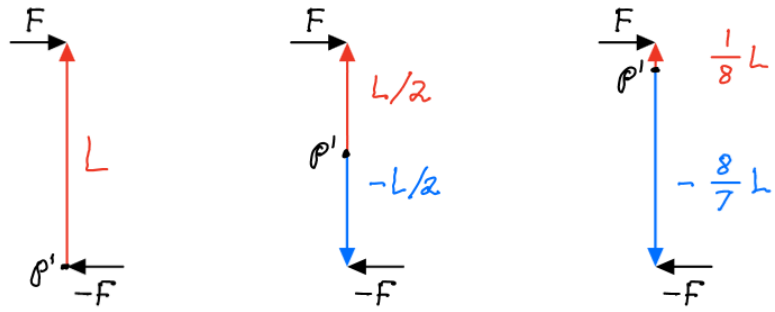
I. MOMENT of FORCE as a $\binom{2}{3}$ TENSOR FIELD (a BIVECTOR-valued 3-FORM)

II. MOMENT of FORCE as TORQUE

Read in MTW §15.3, 15.4, 15.5

(28.1)

As depicted in Figure 28.1, the moment of force on a cube in tranlational equilibrium is independent of the fulcrum location $\mathcal{P}'$.



$$\mathcal{T} = L \wedge F + 0 \wedge (-F) = \frac{L}{2} \wedge F + \frac{-L}{2} \wedge (-F) = \frac{L}{8} \wedge F + \left(-\frac{8}{7} L\right) \wedge (-F)$$

Figure 28.1 The moment of force on a 1-dimensional cube in translational equilibrium is independent of the fulcrum location $\mathcal{P}'$.

I. MOMENT of FORCE as a $\binom{2}{3}$ TENSOR FIELD (a BIVECTOR-valued 2-FORM)

A dielectric 3-cube with no net charge but immersed in a homogeneous electrostatic field experiences the moment of force

$$\overleftrightarrow{\mathcal{T}} = (\mathcal{P}_3^+ - \mathcal{P}_3^-) \wedge \underset{\sim}{\vec{F}}(\vec{u}, \vec{v}) + (\mathcal{P}_2^+ - \mathcal{P}_2^-) \wedge \underset{\sim}{\vec{F}}(\vec{t}, \vec{u}) + (\mathcal{P}_1^+ - \mathcal{P}_1^-) \wedge \underset{\sim}{\vec{F}}(\vec{v}, \vec{t}). \tag{28.1}$$

Here

28.2

$$\left.\begin{aligned} P_3^+ - P_3^- &\equiv \vec{t} = \Delta x^3\, e_3 = e_j \langle dx^j, \vec{t} \rangle \\ P_2^+ - P_2^- &\equiv \vec{v} = \Delta x^2\, e_2 = e_i \langle dx^i, \vec{v} \rangle \\ P_1^+ - P_1^- &\equiv \vec{u} = \Delta x^1\, e_1 = e_m \langle dx^m, \vec{u} \rangle \end{aligned}\right\} \qquad (28.2)$$

are the displacement vectors that separate the opposing faces of the dielectric 3-cube.

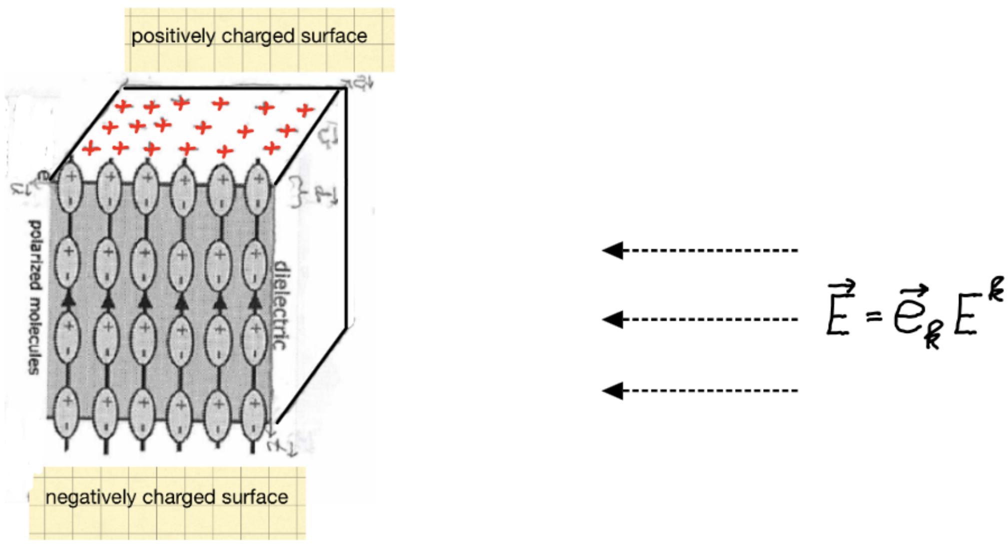


Figure 28.2 Dielectric 3-cube immersed in a homogeneous electrostatic field. The separation between each pair of opposing faces is given by the respective vectors $\vec{u}$, $\vec{v}$, and $\vec{t}$ exhibited by Eq. (28.2).

(28.3)

The introduction of Eq. (28.2) into Eq. (28.1) leads to a non-trivial simplification in the expression for the moment of force, Eq. (28.1).

First of all, recall that all paired forces on the opposing faces of the 3-cube are obtained by evaluating Eq.(26.1'), the surface force density 2-form

$$\underset{\sim}{\vec{F}} = \vec{F}_{ij}\, dx^i \wedge dx^j/2! = e_k E^k q N r^m \epsilon_{mij}\, dx^i \wedge dx^j/2! \qquad (28.3)$$

on the appropriate pair of spanning vectors $(\vec{u},\vec{v})$, $(\vec{v},\vec{t})$, and $(\vec{t},\vec{u})$. Consequently, the expression for the moment of force the 3-cube is subjected to is

$$\overleftrightarrow{\mathcal{T}}(\vec{u},\vec{v},\vec{t}) = \underbrace{\Delta x^3 e_3}_{\vec{t}} \wedge \vec{F}_{|ij|}\, dx^i \wedge dx^j(\vec{u},\vec{v})$$

$$+ \underbrace{\Delta x^2 e_2}_{\vec{v}} \wedge \vec{F}_{|ij|}\, dx^i \wedge dx^j(\vec{t},\vec{u})$$

$$+ \underbrace{\Delta x^1 e_1}_{\vec{u}} \wedge \vec{F}_{|ij|}\, dx^i \wedge dx^j(\vec{v},\vec{t}) \qquad (28.4)$$

Secondly, in spite of superficial appearances to the contrary, the expression for the total moment of force, Eq.(28.4), is a coordinate frame invariant. Indeed, using the basis expansions

(28.4).

$$e_3 \Delta X^3 = e_m \langle dx^m, \vec{t} \rangle$$
$$e_2 \Delta X^2 = e_m \langle dx^m, \vec{v} \rangle$$
$$e_1 \Delta X^1 = e_m \langle dx^m, \vec{u} \rangle$$

Eq. (28.4) becomes

$$\overleftrightarrow{\underset{\sim}{T}}(\vec{u},\vec{v},\vec{t}) = e_m \wedge \vec{F}_{|ij|} \langle dx^m, t \rangle \, dx^i \wedge dx^j (\vec{u},\vec{v})$$
$$+ e_m \wedge \vec{F}_{|ij|} \langle dx^m, \vec{v} \rangle \, dx^i \wedge dx^j (\vec{t},\vec{u}) \rightarrow \text{Their sum equals}$$
$$+ e_m \wedge \vec{F}_{|ij|} \langle dx^m, \vec{u} \rangle \, dx^i \wedge dx^j (\vec{v},\vec{t}) \qquad dx^m \wedge dx^i \wedge dx^j (\vec{u},\vec{v},\vec{t})$$

$$= e_m \, dx^m \wedge \vec{F}_{|ij|} \, dx^i \wedge dx^j (\vec{u},\vec{v},\vec{t}),$$

which in terms of Cartan's unit tensor / "displacement vector"

$$d\mathcal{P} = e_m \, dx^m$$

is

$$\overleftrightarrow{T}(\vec{u},\vec{v},\vec{t}) = d\mathcal{P} \wedge \underset{\sim}{\vec{F}}(\vec{u},\vec{v},\vec{t})$$

or

$$\boxed{\begin{aligned} \overleftrightarrow{\underset{\sim}{T}} &= d\mathcal{P} \wedge \vec{F}_{ij} \, dx^i \wedge dx^j / 2! \\ &= e_m \wedge \vec{F}_{ij} \, dx^m \wedge dx^i \wedge dx^j / 2! \end{aligned}} \qquad (28.6)$$

III. Moment of Force as Torque

(28.5)

The familiar representation of torque is in terms of the vector cross-product

$$\vec{T} = \vec{R} \times \vec{F}.$$

However, the moment of force density

$$\overleftrightarrow{T} = e_m dx^m \wedge \vec{F}_{|ij|} dx^i \wedge dx^j$$

$$= e_m \wedge e_k E^k qN r^n \epsilon_{n|ij|} dx^n \wedge dx^i \wedge dx^j \qquad (28.6)$$

evaluated on the volume of the 3-cube spanned by the triad of vectors $\vec{u}$, $\vec{v}$, and $\vec{t}$ is a bivector.

In spite of their difference, the two representations agree on one key aspect: they are linear spaces with the same dimension,

$$\dim \Lambda^2(E^3) = \dim(E^3).$$

Their bases are

$$\{e_m \wedge e_k : {}^m_k = 1,2,3\} \subset \Lambda^2(E^3)$$

and

$$\{e_\ell : \ell = 1,2,3\} \subset E^3.$$

Thus there exists an isomorphism $\star$ (a special case of the "Hodge duality" mapping),

$$\star : \Lambda^2(E^3) \longrightarrow E^3$$

$$e_m \wedge e_k \rightsquigarrow \star(e_m \wedge e_k) = e_\ell \epsilon^\ell{}_{mk}$$

$$= e_\ell g^{\ell n} [n m k] \sqrt{g}$$

(28.6)

Apply this $\star$ tranformation to Eq. (28.6), a bivector-valued 3-form. The result is the vector-valued 3-form

$$\underset{\sim}{\vec{\mathcal{T}}} = \star(\underset{\sim}{\overleftrightarrow{\mathcal{T}}}) = \star\left(e_m \wedge e_k E^k q N r^n \epsilon_{n|ij|}\, dx^m \wedge dx^i \wedge dx^j\right)$$

$$= e_\ell \in^\ell{}_{mk} E^k q N \; r^n \underbrace{\epsilon_{n|ij|}\,[m\, i\, j]}_{\sqrt{g}\;\delta^m_n}\, dx^1 \wedge dx^2 \wedge dx^3$$

which reduces to

$$= e_\ell \underbrace{\epsilon^{\ell m k}}_{\frac{1}{\sqrt{g}}[\ell m k]} E_k r_m q N \underbrace{\sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3}_{\text{coordinate invariant}}$$

Evaluate this 3-form on the triad of spanning vectors $(\vec{u}, \vec{v}, \vec{t})$ and obtain

$$= \underbrace{\frac{1}{\sqrt{g}} \begin{vmatrix} e_1 & e_2 & e_3 \\ r_1 & r_2 & r_3 \\ qE_1 & qE_2 & qE_3 \end{vmatrix}}_{\vec{r} \times \vec{F}} \underbrace{N \cdot \begin{pmatrix} \text{volume} \\ \text{spanned} \\ \text{by } \vec{u}, \vec{v}, \vec{t} \end{pmatrix}}_{\#\text{ of dipoles}}$$

This is the moment of force suffered by # dipoles,

$$\# = N \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3 \,(\vec{u}, \vec{v}, \vec{t}),$$

each subjected to the torque

$$\vec{r} \times \vec{F} = \frac{1}{\sqrt{g}} \begin{vmatrix} e_1 & e_2 & e_3 \\ r_1 & r_2 & r_3 \\ qE_1 & qE_2 & qE_3 \end{vmatrix} .$$

Thus the moment of force applied to the faces of a (28.7)
dielectric 3-cube physically equals (modulo the Hodge isomorphism $\star$) the sum total torque applied to each and everyone of the molecular dipoles occupying the volume of the 3-cube:

$$\underset{\sim}{\overleftrightarrow{\mathcal{T}}}(\vec{u},\vec{v},\vec{t}\,) \overset{\star}{\approx} \overrightarrow{\mathcal{T}}(\vec{u},\vec{v},\vec{t}\,) = \star\left(\left(\vec{r}\times\vec{F}\right)\cdot\#\right)$$

The face representation $\underset{\sim}{\overleftrightarrow{\mathcal{T}}}$ of the stressed dielectric 3-cube is related to its volume representation $\underset{\sim}{\overrightarrow{\mathcal{T}}}$ by means of the Hodge isomorphism $\star$.