Lecture 29

Electric Field induced Moment of Force
vs
Curvature induced Moment of Rotation

I. Moment of Force vs. Torque

A. Moment of force

B. Moment of rotation

II. Rotational "$\vec{F} = m\vec{a}$"

vs

Einstein Field Equation

III Two equivalent momenergy representations

In MTW read all of Ch. 15.

I. Moment of Force vs. Torque

29.1

A homogeneous electrostatic field subjects a polarized dielectric 3-cube to a moment of force, which is mathematized by the bivectorial 3-form

$$\underset{\sim}{\overset{\leftrightarrow}{\mathcal{T}}} = d\mathcal{P}\wedge \underset{\sim}{\vec{F}} = e_\ell \wedge e_k F^{\;k}_{|ij|}\, dx^\ell \wedge dx^i \wedge dx^j$$

$$= e_\ell \wedge e_k E^k q N r^m \epsilon_{m|ij|}\, dx^i \wedge dx^j$$

Here, we recall, that

$$\vec{r} = e_m r^m$$

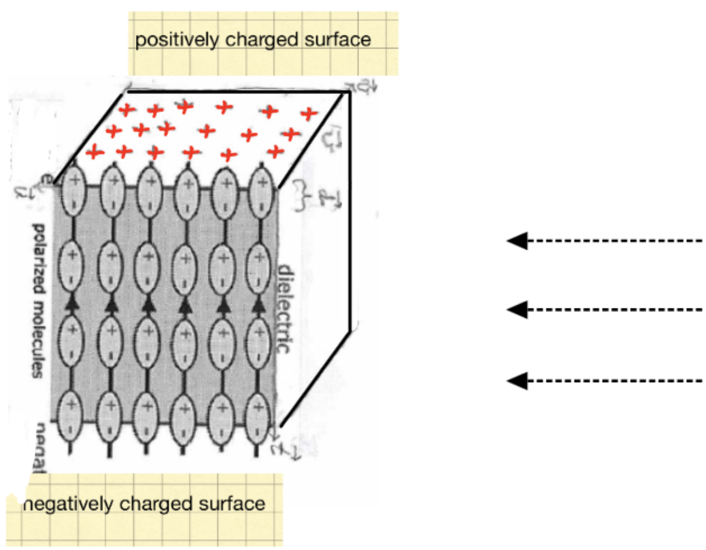
is the vector displacement that separates the positive from the negative charge of a molecular dipole $\vec{p} = q\vec{r}$ so that

$$q N r^m \epsilon_{m|ij|}\, dx^i \wedge dx^j (\vec{u},\vec{v}) = \left(\begin{array}{l}\text{amount of charge on} \\ \text{the face spanned by} \\ \text{the vectors } \vec{u} \text{ and } \vec{v}.\end{array}\right)$$

With the opposing face having opposite charge, the dipole moment is

$$\vec{r} q N r^m \epsilon_{m|ij|}\, dx^i \wedge dx^j (\vec{u},\vec{v}) \equiv \vec{p}$$

and the corresponding torque on the 3-cube is $\vec{p}\times\vec{E}$. Additional contribution come from the other opposing faces.



$$\vec{E} = \vec{e}_k E^k$$

Figure 29.1 The surface charges on the boundary of a polarized dielectric 3-cube interacting with the electrostatic field $\vec{E}$ cause the 3-cube to be subjected to a moment of force. (29.2)

Although the cause of the moment of force is confined to the 3-cube's surface, its magnitude is proportional to the volume, and hence to the number of molecular dipole moments $q\vec{r}$ of the 3-cube. Indeed, the vectorial torque on the 3-cube spanned by the 3-d vectors $\vec{u}$, $\vec{v}$, and $\vec{E}$ is

$$\vec{\underset{\sim}{T}}(\vec{u},\vec{v},\vec{E}) = \star\left(\overleftrightarrow{\underset{\sim}{T}}(\vec{u},\vec{v},\vec{E})\right) = \underbrace{\frac{1}{\sqrt{g}}\begin{vmatrix} e_1 & e_2 & e_3 \\ r_1 & r_2 & r_3 \\ qE_1 & qE_2 & qE_3 \end{vmatrix}}\;\underbrace{N\times\begin{pmatrix}\text{volume}\\ \text{spanned}\\ \text{by } \vec{u},\vec{v},\vec{E}\end{pmatrix}}$$

$$= \left(\vec{r}\times q\vec{E}\right)\times\left(\#\text{ of dipoles}\right)$$

In 3-d Euclidean space there are two physically equivalent ways of mathematizing the stressed state of a polarizable dielectric 3-cube:

1. Via the bivector-valued moment of force representation

$$\overleftrightarrow{\underset{\sim}{\mathcal{T}}} = d\mathcal{P} \wedge \underset{\sim}{\vec{F}} = e_\ell \wedge e_k F^{k}_{|ij|}\, dx^\ell \wedge dx^i \wedge dx^j \tag{29.3}$$

and

2. via the vector-valued torque representation

$$\underset{\sim}{\vec{\mathcal{T}}}(\vec{u},\vec{v},\vec{t}) = \underbrace{\frac{1}{\sqrt{q}}\begin{vmatrix} e_1 & e_2 & e_3 \\ r_1 & r_2 & r_3 \\ qE_1 & qE_2 & qE_3 \end{vmatrix}}_{=\ (\vec{r}\times q\vec{E})} \underbrace{N \times \begin{pmatrix}\text{volume} \\ \text{spanned} \\ \text{by } \vec{u},\vec{v},\vec{t}\end{pmatrix}}_{\times\ (\#\text{ of dipoles})} .$$

However, for the purpose of generalization to 4-d spacetime and applying it to the Einstein field equations, only the method of moments leads to a geometrically valid formulation.

II. Spacetime Moment of Rotation.

The generalization from Euclidean 3-space to 4-d spacetime consists of extending the concept of an electrostatic field-induced moment of polarization vector to that of a gravitational field-induced moment of rotation bivector.

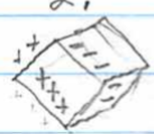
I. B) 4-d SPACETIME

29.4

The generalization to 4-d space is now straight and proceeds as follows:

1. 3-cube in E^3 | 1. 3-cube in 4-d spacetime

2. Vector-valued force field density | 2. Bivector-valued, curvature-induced rotation field density



$$\underset{\sim}{\vec{F}} = \vec{e}_k F^k{}_{|ij|}\, dx^i \wedge dx^j \qquad \underset{\sim}{\overleftrightarrow{R}} = e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{|\alpha\beta|}\, dx^\alpha \wedge dx^\beta$$

3. Translational equilibrium | 3. Bianchi identity

a) $\sum_{\ell=1}^{6} \vec{F}(\ell^{th}\text{ face}) = 0$ | a) $\sum_{\ell=1}^{6} \overleftrightarrow{R}(\ell^{th}\text{ face}) = 0$

b) $d\underset{\sim}{\vec{F}} = 0$ | b) $d\overleftrightarrow{R} = 0$

c) $F^k{}_{[ij;n]} = 0$ | c) $R^{\lambda\mu}{}_{[\alpha\beta;\gamma]} = 0$

4. Moment of Force/(volume) | Moment of Rotation / volume

$$\underset{\sim}{\overleftrightarrow{\mathcal{T}}} = d\mathcal{P} \wedge \underset{\sim}{\vec{F}} \qquad d\mathcal{P} \wedge \overleftrightarrow{R} =$$

$$= e_\ell \wedge e_k F^k{}_{|ij|}\, dx^\ell \wedge dx^i \wedge dx^j \qquad = e_\nu \wedge e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{|\alpha\beta|}\, dx^\nu \wedge dx^\alpha \wedge dx^\beta$$

$$\bigstar(d\mathcal{P} \wedge \overleftrightarrow{R}) = e_\sigma \epsilon_{\nu\lambda\mu}{}^{\sigma} R^{|\lambda\mu|}{}_{|\alpha\beta|}\, dx^\nu \wedge dx^\alpha \wedge dx^\beta = \frac{8\pi G}{c^2} * \vec{\mathbb{T}}$$

$$\vec{\mathcal{T}} = \bigstar(\overleftrightarrow{\mathcal{T}})$$

$$= e_m \epsilon^m{}_{\ell k} F^k{}_{|ij|}\, dx^\ell \wedge dx^i \wedge dx^j \qquad = \frac{8\pi G}{c^2} e_\sigma T^{\sigma\tau} \epsilon_{\tau|\alpha\beta\gamma|}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma$$

II. EINSTEIN'S FIELD EQ'N

29.5

A.) The geometrical statement of the Einstein field equations is

$$\boxed{\frac{\begin{pmatrix}\text{Moment of}\\ \text{rotation}\end{pmatrix}}{\begin{pmatrix}\text{Spacetime}\\ \text{3-volume}\end{pmatrix}} = \frac{8\pi G}{c^4}\,\frac{(\text{Momenergy})}{\begin{pmatrix}\text{Spacetime}\\ \text{3-volume}\end{pmatrix}}}$$

The mathematized version of this statement is

$$\boxed{d\mathcal{P}\wedge\overleftrightarrow{\mathcal{R}} = \frac{8\pi G}{c^2}\,\star^{-1}(*\mathbb{T})}$$

or, equivalently

$$\boxed{\star(d\mathcal{P}\wedge\overleftrightarrow{\mathcal{R}}) = \frac{8\pi G}{c^2}*\mathbb{T}}$$

The momenergy-valued 3-form is a <u>bulk</u> property.

III. Momenergy / volume has two equivalent representation

1.) As a vector-valued 3-volume density

$$\star\mathbb{T} = e_\rho T^{\rho\sigma} \underbrace{\epsilon_{\sigma\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma}_{d^3\Sigma_\sigma = (\text{inv't vol})_\sigma}$$

and

2.) as a tri-vector valued 3-volume density

$$\star^{-1}(\star\mathbb{T}) = e_\nu \wedge e_\lambda \wedge e_\mu \epsilon^{\nu\lambda\mu}{}_\rho T^{\rho\sigma} d^3\Sigma_\sigma$$

This equivalence is based on the isomorphism between two 4-dimensional linear spaces

$$\begin{bmatrix}\star \\ \star^{-1}\end{bmatrix} : R^4 \wedge R^4 \wedge R^4 \underset{\star^{-1}}{\overset{\star}{\rightleftarrows}} R^4$$

$$e_\nu \wedge e_\lambda \wedge e_\mu \rightsquigarrow \star(e_\nu \wedge e_\lambda \wedge e_\mu) = \epsilon_{\nu\lambda\mu}{}^\rho e_\rho$$

$$e_\rho \rightsquigarrow \star^{-1}(e_\rho) = \frac{1}{3!} e_\nu \wedge e_\lambda \wedge e_\mu \epsilon^{\nu\lambda\mu}{}_\rho$$

$$\star\star^{-1}(e_\rho) = e_\rho$$

B.) Component formulation of the E.F.E.:

$$d\mathcal{P}\wedge \overset{\leftrightarrow}{\mathcal{R}} = e_\nu \wedge e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{|\alpha\beta|}\, dx^\nu \wedge dx^\alpha \wedge dx^\beta = \frac{8\pi G}{c^4} \star^{-1}(* \mathbb{T})$$

$$= \frac{8\pi G}{c^4} \frac{e_\nu\wedge e_\lambda \wedge e_\mu}{3!} \epsilon^{\nu\lambda\mu}{}_\rho T^{\rho\sigma} d^3\Sigma_\sigma$$

$$T^{\rho\sigma} e_\sigma \cdot e_\rho \underbrace{\epsilon^\rho{}_{\alpha\beta\gamma}\, dx^\alpha\wedge dx^\beta \wedge dx^\gamma}_{d^3\Sigma^\rho}$$

OR equivalently

$$\star\left(d\mathcal{P}\wedge \overset{\leftrightarrow}{\mathcal{R}}\right) = \epsilon_{\nu\lambda\mu}{}^\rho e_\rho R^{|\lambda\mu|}{}_{|\alpha\beta|}\, dx^\nu\wedge dx^\alpha \wedge dx^\beta$$

$$= \frac{8\pi G}{c^4} e_\rho T^{\rho\sigma} d^3\Sigma_\sigma$$

Introduce

$$R^{\delta\mu}{}_{\delta\nu} \equiv R^\mu{}_\nu \qquad \text{(Ricci)}$$

$$R^\delta{}_\delta = R \qquad \text{(Curvature invt)}$$

and obtain

$$\boxed{R^\mu{}_\nu - \tfrac{1}{2}\delta^\mu{}_\nu R = \frac{8\pi G}{c^4} T^\mu{}_\nu}$$

APPENDIX TO LECTURE 29

5.) Hodge dual on Euclidean space E^3

a) $\star: \Lambda^2(E^3) \to E^3$

$$e_\ell \wedge e_k \rightsquigarrow \star(e_\ell \wedge e_k) = e_n \epsilon^n{}_{\ell k}$$

$$\star\left(d\mathcal{P} \wedge \underset{\sim}{\vec{F}}\right) = \vec{e}_n \epsilon^n{}_{\ell k} F^k{}_{|ij|}\, dx^\ell \wedge dx^i \wedge dx^j$$

b) Inverse Hodge dual

$$\star^{-1}: e_m \rightsquigarrow \star^{-1}(e_m) = \frac{1}{2!} e_\ell \wedge e_k \epsilon^{\ell k}{}_m$$

c) $\star\star^{-1}(e_m) = \frac{1}{2!} e_n \epsilon^n{}_{\ell k}\, \epsilon^{\ell k}{}_m$

$$= e_n \delta^n{}_m \text{ (identity!)}$$

d) $\star^{-1}\star(e_\ell \wedge e_k) = \star^{-1}(e_n \epsilon^n{}_{\ell k})$

$$= \frac{1}{2!} e_i \wedge e_j \epsilon^{ij}{}_n \epsilon^n{}_{\ell k}$$

$$= \frac{1}{2!} e_i \wedge e_j \delta^{ij}_{\ell k}$$

$$= e_\ell \wedge e_k \text{ (identity!)}$$

Hodge dual on Minkowski spacetime R^4

a) $\star: \Lambda^3(R^4) \to R^4$

$$e_\nu \wedge e_\lambda \wedge e_\mu \rightsquigarrow \star(e_\nu \wedge e_\lambda \wedge e_\mu) = \epsilon_{\nu\lambda\mu}{}^\sigma e_\sigma$$

$$\star\left(d\mathcal{P} \wedge \underset{\sim}{\overleftrightarrow{R}}\right) = \epsilon_{\nu\lambda\mu}{}^\sigma e_\sigma R^{\lambda\mu}{}_{|\alpha\beta|}\, dx^\nu \wedge dx^\alpha \wedge dx^\beta$$

b) Inverse Hodge dual

$$\star^{-1}: e_\rho \rightsquigarrow \star^{-1}(e_\rho) = \frac{-1}{3!} e_\nu \wedge e_\lambda \wedge e_\mu \epsilon^{\nu\lambda\mu}{}_\rho$$

c) $\star\star^{-1}(e_\rho) = \frac{-1}{3!} \epsilon_{\nu\lambda\mu}{}^\sigma e_\sigma\, \epsilon^{\nu\lambda\mu}{}_\rho$

$$= (+)\delta^\sigma_\rho e_\sigma \text{ (identity!)}$$

d) $\star^{-1}\star(e_\nu \wedge e_\lambda \wedge e_\mu) = \star^{-1}(\epsilon_{\nu\lambda\mu}{}^\sigma e_\sigma)$

$$= \epsilon_{\nu\lambda\mu}{}^\sigma \frac{-1}{3!} e_\alpha \wedge e_\beta \wedge e_\gamma \epsilon^{\alpha\beta\gamma}{}_\sigma$$

$$= \frac{1}{3!} e_\alpha \wedge e_\beta \wedge e_\gamma \delta^{\alpha\beta\gamma}_{\nu\lambda\mu}$$

$$= e_\nu \wedge e_\lambda \wedge e_\mu \text{ (identity!)}$$