LECTURE 3

World lines of extremal length

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(2.11)

V. PARAMETRIZATION INVARIANCE

1.) Noether's Theorem

Before engaging in a mathematical frontal attack on Eqs. (2.9), it is extmely rewarding to take note of the fact that the variational integral, Eq(2.3), is invariant under a transformation of its curve parameter. This invariance permits one to master the apparent complexity of Eqs. (2.8) and (2.9).

Indeed, let $\lambda \to \bar\lambda = \bar\lambda(\lambda)$, $\bar\lambda \to \lambda = \lambda(\bar\lambda)$ its inverse, with $\bar\lambda(\lambda=0)=0$ and $\bar\lambda(\lambda=1)=1$. Consider the new reparametrized trajectory $x^\alpha = \bar a^\alpha(\lambda) = a^\alpha(\bar\lambda(\lambda))$ (2.10) instead of the old one,

$$x^\alpha = a^\alpha(\lambda). \qquad (2.11)$$

The value of the variational integral Eq. (2.3) on page 2.7 is

$$\tau_A{}^B[\bar a(\cdot)] = \int_{\lambda=0}^{\lambda=1} \sqrt{-g_{\mu\nu}\left(a^\alpha(\bar\lambda(\lambda))\right) \frac{d a^\mu(\bar\lambda(\lambda))}{d\lambda} \frac{d a^\nu(\bar\lambda(\lambda))}{d\lambda}}\, d\lambda \qquad (2.12)$$

$$= \int_0^1 \sqrt{-g_{\mu\nu}(a^\alpha(\bar\lambda)) \frac{da^\mu(\bar\lambda)}{d\bar\lambda}\frac{d\bar\lambda}{d\lambda}\frac{da^\nu(\bar\lambda)}{d\bar\lambda}\frac{d\bar\lambda}{d\lambda}}\, d\lambda$$

$$= \int_{\bar\lambda=0}^{\bar\lambda=1} \sqrt{-g_{\mu\nu}\left(a^\alpha(\bar\lambda)\right) \frac{d a^\mu(\bar\lambda)}{d\bar\lambda}\frac{d a^\nu(\bar\lambda)}{d\bar\lambda}}\, d\bar\lambda = \tau_A{}^B(a(\cdot)) \qquad (2.13)$$

Thus the value the variational integral is

independent of <u>all</u> worldlines connecting (2.12)
point events A and B in Figure 2.5, including those which are not extremal spactime trajectories.

One says that the integral is <u>invariant</u> <u>under the</u> <u>transformation</u> that takes a^α into $\bar{a}^\alpha$.
This transformation is one which is "global" in that it is a statement about any (extremal or non-extremal) A-to-B trajectory as a <u>whole</u>.
The benefit derived from this observation is that, via variational calculus (which led to Eqs. (2.9)), it implies a differential identity which holds <u>pointwise</u> along <u>any</u> curve in the domain where it is defined. This identity, which is Eq. (2.17) below on page 2.17,

implies the "conservation law"

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)=0, \qquad (2.14)$$

i.e. $g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$ is a constant along <u>any</u> trajectory whenever it is parametrized by the proper time τ.

The transformation invariance of the variational integral Eqs. (2.12)-(2.13) on page 2.11 implies the corresponding "conservation law", Eq. (2.14). This conclusion is an illustration of what is known as <u>Noether's Theorem</u>.

2.) Underdetermined System

(a) The equations $\{f_\gamma(\lambda)=0: \gamma=0,1,2,3\}$, namely Eqs. (2.9) on page 2.10a,

constitute an underdetermined (2.14)
system. For any solution $a^\alpha(\lambda)$ to $f_\gamma(\lambda)=0$ there are many other solutions. Indeed, for an arbitrary function $\lambda(\bar\lambda)$, in Eq. (2.7) on page 2.9 one finds that

$$f_\gamma(\lambda(\bar\lambda)) = f_\gamma(\bar\lambda).$$

Consequently, if $a^\alpha(\lambda)$ is a solution to $f_\gamma(\lambda)=0$, then $a^\alpha(\lambda(\bar\lambda)) \equiv \bar a^\alpha(\bar\lambda)$ is also a solution to $f_\gamma(\bar\lambda)=0$.

(b) The equations $f_\gamma(\lambda)=0$ on page 2.9 constitute <u>mathematical overkill.</u> There are more of them than necessary to express the extremal nature of the variational integral

$$\tau_A{}^B[X^\alpha] = \int_A^B \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda . \quad (2.15)$$

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In other words, one of these equations holds for all world lines, even those that do not extremize $\tau_A{}^B$.
This fact follows from its invariance under reparametrization. The reparametrization

$$\lambda \to \bar\lambda : \ \bar\lambda(\lambda) = \lambda + h(\lambda) \qquad (2.16)$$

subject to

$$\left.\begin{array}{l}\bar\lambda(0)=0\\ \bar\lambda(1)=1\end{array}\right\} \Longleftrightarrow h(0)=h(1)=0$$

does not change the value of the variational integral.

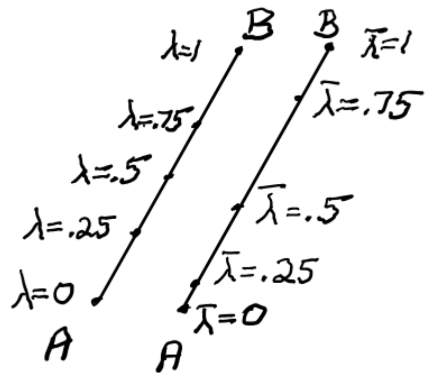

Figure 2.6 Two different parametrizations of the worldline AB.

It corresponds to a mere "repositioning of the beads along a string" (=reparametrization). This process is mathematized by the statement

$$\bar{a}^\alpha(\lambda) = a^\alpha(\bar{\lambda}(\lambda))$$

Apply this to Eq. (2.15) on page 2.15 and find that the value of the variational integral,

$$\begin{aligned}\tau_A{}^B[\bar{a}(\cdot)] &= \int_{\lambda=0}^{\lambda=1} \sqrt{-g_{\mu\nu}(a^\alpha(\bar{\lambda}(\lambda)))\frac{da^\mu(\bar{\lambda}(\lambda))}{d\lambda}\frac{da^\nu(\bar{\lambda}(\lambda))}{d\lambda}}\,d\lambda = \int_{\bar{\lambda}=0}^{\bar{\lambda}=1} \sqrt{-g_{\mu\nu}(a^\alpha(\bar{\lambda}))\frac{da^\mu(\bar{\lambda})}{d\bar{\lambda}}\frac{da^\nu(\bar{\lambda})}{d\bar{\lambda}}}\,d\bar{\lambda} \\ &= \int_{\lambda=0}^{\lambda=1} \sqrt{-g_{\mu\nu}(a^\alpha(\lambda))\frac{da^\mu(\lambda)}{d\lambda}\frac{da^\nu(\lambda)}{d\lambda}}\,d\lambda = \tau_A{}^B(a(\cdot)),\end{aligned}$$

has not changed, even though $\bar{\lambda}(\lambda)$ is not the identity function.

The first order change in $a^\gamma(\lambda)$ brought about by the reparametrization (2.16) on page 2.15 is

$$a^\gamma(\lambda) \longrightarrow a^\gamma(\bar\lambda) = a^\gamma(\lambda + h(\lambda)) = a^\gamma(\lambda) + \delta a^\gamma(\lambda)$$

where

$$\delta a^\gamma(\lambda) = \frac{da^\gamma}{d\lambda} h(\lambda) \qquad \left(\text{"Principal linear part"}\right)$$

and higher order terms have been neglected.

The fact that such variations can not change the variational integral for arbitrary $h(\lambda)$ implies

$$\delta \tau_A{}^B = \int_0^1 f_\gamma(\lambda) \frac{da^\gamma}{d\lambda} h(\lambda) \sqrt{\quad} \, d\lambda = 0$$

Consequently,

$$\boxed{f_\gamma(\lambda) \frac{da^\gamma}{d\lambda} = 0} \quad (\text{even if } f_\gamma \neq 0!) \quad (2.17)$$

This holds for all paths $a^\gamma(\lambda)$, even those that do not extremize $\tau_A{}^B$!

An equation that holds, whether or not the quantities obey any differential equation, is called an identity. Here it is simply an algebraic identity.

VI. PROPER TIME PARAMETRIZATION

The reparametrization freedom is a green light to simplifying the differential equation (2.9) on page 2.10: For any worldline $a^\alpha(\lambda)$ introduce the proper time increment

$$d\tau = \sqrt{-g_{\alpha\beta}(a^\gamma(\lambda))\,\frac{da^\alpha(\lambda)}{d\lambda}\frac{da^\beta(\lambda)}{d\lambda}}\,d\lambda \qquad (2.18)$$

The derivatives $\frac{d\tau}{d\lambda}$ and $\frac{d\lambda}{d\tau}$ are well-defined. Consequently, $\tau(\lambda)$ is a monotonic function and so is its inverse $\lambda(\tau)$. Using proper time τ as the new parameter, introduce

$$a^\gamma(\lambda(\tau)) \equiv x^\gamma(\tau). \qquad (2.19)$$

Thus

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$$\frac{dx^\gamma(\tau)}{d\tau}=\frac{da^\gamma(\lambda)}{d\lambda}\frac{d\lambda}{d\tau}=\frac{1}{\sqrt{\quad}}\frac{da^\gamma}{d\lambda}\equiv\dot{x}^\gamma(\tau)\qquad(2.20)$$

where $\sqrt{\quad}$ is the non-zero square root expression in Eq.(2.18).

In terms of proper time as the new curve parameter, the extremum condition as expressed by Eq. (2.8) and (2.9) on page 2.10 simplifies enormously:

$$0=f_\gamma(\lambda)=\frac{1}{2}\frac{d}{d\tau}\left(g_{\gamma\nu}\frac{dx^\nu}{d\tau}\right)+\frac{1}{2}\frac{d}{d\tau}\left(g_{\mu\gamma}\frac{dx^\mu}{d\tau}\right)-\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\gamma}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

or*

$$0=g_{\gamma\nu}\frac{d^2x^\nu}{d\tau^2}+\frac{1}{2}\left(g_{\gamma\nu,\mu}+g_{\gamma\mu,\nu}-g_{\mu\nu,\gamma}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\qquad(2.21)$$

*\footnote{ Here and elsewhere $g_{\gamma\nu,\mu}$ is short hand for the partial derivative

$$g_{\gamma\nu,\mu}\equiv\frac{\partial g_{\gamma\nu}}{\partial x^\mu}.\}$$

Streamline this differential equation further by introducing the inverse metric $g^{\alpha\gamma}$:

$$g^{\alpha\gamma} g_{\gamma\sigma} = \delta^{\alpha}_{\sigma}. \qquad (2.20)$$

Apply it to Eq. (2.21) and obtain

$$\boxed{0 = \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^{\alpha}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \qquad (2.22)$$

$$\boxed{0 = \ddot{x}^\alpha + \Gamma^{\alpha}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu},$$

where $\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\gamma} (g_{\gamma\mu,\nu} + g_{\gamma\nu,\mu} - g_{\mu\nu,\alpha})$ (2.23) is the "Chriffel symbol of the 2nd kind."**

** \footnote{ By contrast, the "Christoffel symbols of the 1st kind" are the ones in Eq. (2.21),

$$\Gamma_{\gamma\mu\nu} = \frac{1}{2}(g_{\gamma\mu,\nu} + g_{\gamma\nu,\mu} - g_{\mu\nu,\gamma}),$$

before we introduced the inverse metric. These symbols are important in that they mathematize the metric compatibility of the law of parallel transport.

Indeed, add to the above symbol the one with γ and ν interchanged:

$$\Gamma_{\nu\mu\gamma} = \frac{1}{2}\left(g_{\nu\mu,\gamma} + g_{\gamma\nu,\mu} - g_{\mu\gamma,\nu}\right).$$

Their sum is

$$\frac{\partial g_{\gamma\nu}}{\partial x^\mu} = \Gamma_{\gamma\mu\nu} + \Gamma_{\nu\mu\gamma}$$

$$= g_{\gamma\alpha}\Gamma^\alpha{}_{\mu\nu} + g_{\alpha\nu}\Gamma^\alpha{}_{\mu\gamma}\,,$$

$$0 = \frac{\partial g_{\gamma\nu}}{\partial x^\mu} - g_{\gamma\alpha}\Gamma^\alpha{}_{\nu\mu} - g_{\alpha\nu}\Gamma^\alpha{}_{\mu\gamma} \equiv g_{\gamma\nu;\mu}$$

The right hand side are the components of the covariant derivative of the metric tensor. That they vanish expresses the fact the metric is <u>covariantly constant</u>. This is the condition that the law as expressed by the $\Gamma^\alpha{}_{\mu\nu}$ in Eq. (2.23) on page 2.20 is

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(a) compatible with the same metric that went into the extremization as stated on page 2.5, and

(b) has zero torsion (because $\Gamma^{\alpha}_{\mu\nu}=\Gamma^{\alpha}_{\nu\mu}$)

The line of reasoning leading to the equation for a geodesic lead to two conclusions:

① The principle of extremal proper time implies a unique torsionless parallel transport which is compatible with the metric:

$$\int\sqrt{-g_{\mu\nu}\,dx^{\mu}dx^{\nu}} = \text{extr.} \begin{cases} \Rightarrow \Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\gamma}\left(g_{\gamma\mu,\nu} + g_{\gamma\nu,\mu} - g_{\mu\nu,\gamma}\right) \\ \Rightarrow \Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu} \quad (\text{i.e. torsion}=0) \end{cases}$$

② The geodesics of curved spacetime coincide with the worldlines of extremal proper time.

VII. A GENERAL CONSTANT OF MOTION

By differentiating the squared magnitude $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ w.r.t. τ, one can verify that

$$f_\gamma(\lambda)\frac{dx^\gamma}{d\tau}=0 \Rightarrow \frac{d}{d\tau}\left(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right)=0$$

Thus $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \text{const}$ ($=-1$ for <u>any</u> time-like curve). $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ is <u>always</u> an integral of motion; it expresses the constancy of the magnitude of the unit tangent $\mathbf{u}=\frac{dx^\mu}{d\tau}\frac{\partial}{\partial x^\mu}$:

$$\mathbf{u}\cdot\mathbf{u} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \text{const.}$$

for any curve, even if it is not a geodesic.