Lecture 30

EFEs: Moment of Rotation per 3-volume = Momenergy per 3-volume

I. The Einstein Field Equations.

Einstain started his process of mathematizing gravitation in 1907 when he introduced two fundamental concepts: (1) An accelerated frame as a one-parameter family of instantaneous inertial ("free float") frames, mathematicians nowadays call "tangent spaces," and (2) his equivalence principle, according to which the behavior of things in a uniformly accelerated frame, e.g. the free motion of bodies in a rocket), is indistinguishable from the behavior of things in a local uniform gravitation field; inother words "inertial" forces are indistinguishable of forces due to gravition.

He took his next fundamental step in 1913 when he realized that, to mathematize (a) the source of gravitation and (b) the motion of bodies under its influence, one must do so in geometrical terms, namely using the methods developed by Gauss, Riemann, Ricci, Levi-Civita, and others.

By a subsequent tour de force he arrived in 1915 at his field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu}. \qquad (30.1)$$

His line of reasoning was based on his 1913 recognition that his equations had to be tensorial in nature. His guiding principle was based on the nature and the conservation laws of the r. h. s. of his equation, the momentum and energy ("momenergy") tensor, which is the source of the gravitational field.

Indeed, by applying his 1913 recognition to the Poisson equation for the Newtonian gravitational potential,

$$\nabla^2 \phi_{NEWTON} = 4\pi G \rho$$

he made three inferences:

(i) from his mass energy relation, he inferred that the source of that equation,

$$\nabla^2 \phi_{NEWTON} = \frac{4\pi G}{c^2} \rho c^2$$

is the mass-energy density, which is only part of the momentum-energy ("momenergy") tensor, and

(ii) from the geometrization of Newton's first law of motion relative to non-inertial reference frames plus his equivalence priciple, he inferred that, for weak gravitational fields, the Newtonian potential ϕ_{NEWTON} is related (see Eq. (5.9) on page 5.13) to the g_{00} component of the space time tensor

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

by the equation

$$g_{00} = -1 - \frac{2G}{c^2}\phi_{NEWTON}.$$

(iii) from the fact that the only tensor involving the 2^{nd} derivatives of the metric tensor is the Riemann-Christoffel curvature tensor, he inferred that what he must look for is a tensorial equation equation for the components of the metric, based on the curvature tensor with the momergy tensor as the source.

Its a consequence of these three inferences the tensorial l.h.s. of his equations was a mathematically deductive consequence. As to its geometrical and physical nature, that was left in a shroud of mystery until E. Cartan in 1925 from a geometrical perspective, and J.A. Wheeler in 1964 (with help from his student C.W. Misner) from a physics perspective, identify the l.h.s. of Eq.(30.1) as the moment of curvature-induced rotation. Because of this, the E.F.Eq'ns state a causal relationship between gravitation and mater:

For any given volume element in spacetime,

$$\text{Moment of rotation} = \frac{8\pi G}{c^2}\,\text{Momenergy}$$

or

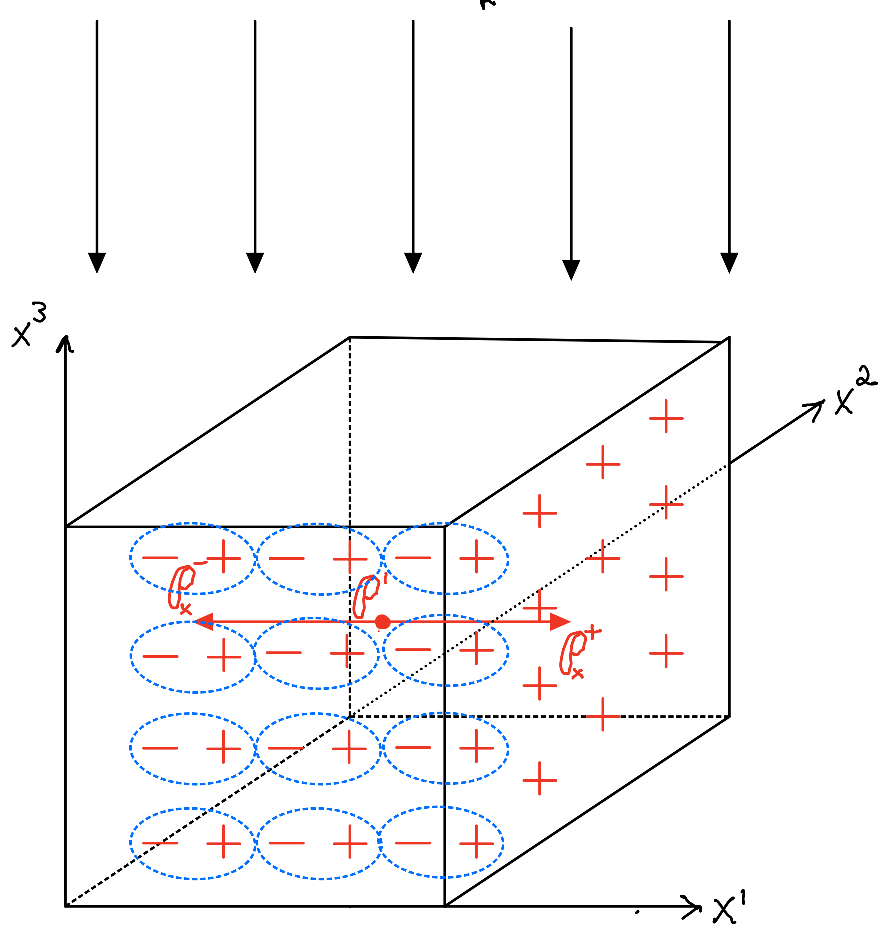
$$\frac{\begin{pmatrix}\text{Moment}\\ \text{of rotation}\end{pmatrix}}{\begin{pmatrix}\text{Spacetime}\\ \text{3-volume}\end{pmatrix}} = \frac{8\pi G}{c^2}\,\frac{(\text{Momenergy})}{\begin{pmatrix}\text{Spacetime}\\ \text{3-volume}\end{pmatrix}} \qquad (30.2)$$

The line of reasoning leading to this statement of the E.F.E.s is to start with well-known concepts from mechanics and electrostatic, mathematize them in term of differential forms, and then extend them from 3-d Euclidean space to the 4-d space-time.

II. Moment of Electrostatic Force

A dielectric with non-zero polarization when immersed into the force field of a homogeneous electrostatic

$$\vec{E} = e_k E^k$$



30.6

Figure 30.1 As depicted, a force couple acts on a pair of oppositely charged opposing faces of a polarized dielectric cube immersed in the homogeneous electrostatic field $\vec{E} = e_k E^k$.
The vectorial separation $\mathcal{P}^+ - \mathcal{P}^-$ is a lever arm. Together with the two opposite forces of the couple, it furnishes the moment of force, which tends to make the cube rotate.

field, as depicted in Figure 30.1, gets subjected to a shear force couple. It acts on the oppositly charged areas of the opposing pair of faces. These two forces are mathematized by evaluating the vectorial surface force density $\left(\frac{(\text{force})}{(\text{area})}\right)$

$$\underset{\sim}{\vec{F}} = e_k F^k_{|ij|}\, dx^i \wedge dx^j = e_k E^k q N r^\ell \epsilon_{\ell|kj|}\, dx^i \wedge dx^j \qquad (30.3)$$

on each of the two faces.
The displacement vector

$$\mathcal{P}^+_x - \mathcal{P}^-_x = d\mathcal{P}(e_1 \Delta x^1) = e_\ell\, dx^\ell(e_1 \Delta x^1) = e_\ell \langle dx^\ell | e_1 \rangle \Delta x^1 = e_1 \Delta x^1,$$

which separates the two faces, or — for that matter separates any other pair of faces, is mathematized by Cartan's "unit tensor"

$$d\mathcal{P} = e_\ell\, dx^\ell = e_\ell\, \delta^\ell_j \otimes dx^j \qquad (30.4)$$

This vectorial 1-form is a vector which refers to an as-yet-unspecified displacement away from its fulcrum. Together with the force, Eq. (30.3) it forms a new mathematical concept*, namely that of the moment of force, (30.7)

*\footnote{The formation of the concept "moment of force" is illustrate pictorially in Figure 28.1. For a philosophically precise explanation on how to form a concept see Chapter 2 in "Introduction to Objectivist Epistemology" by Ayn Rand. The "Conceptual Common Denominator" in that chapter is all instances of the moment of force possess a fulcrum as a common feature, fulcrum whose particular location exists but can be omitted from explicit reference to the concept "moment of force." This irrelevance of the particular of the particular location of the fulcrum is illustrated very graphically in Figure 27.3 and 28.1 and is also known as the principle of measurement omission in the theory of concept formation.

$$\underset{\sim}{\overleftrightarrow{\mathcal{T}}} = d\mathcal{P} \wedge \underset{\sim}{\vec{F}} \tag{30.5}$$

$$= e_\ell\, dx^\ell \wedge \underset{\sim}{\vec{F}} = e_\ell \wedge e_k\, F^k_{\ |ij|}\, dx^\ell \wedge dx^i \wedge dx^j \tag{30.6}$$

This is a bivector-valued 3-form. Evaluate it on the three vectors that span the volume of the 3-cube depicted in Figure 30.1. The result is a linear combination of bivectors. They are elements of a linear space which is 3-dimensional. It follows that this space is isomorphic to a 3-dimensional space of vectors. There is a one-to-one correspondence between bivectors and vectors. This correspondence ("isomorphism") is unique. It is a special case of the Hodge duality map. It maps elements of area spanned by a pair of 3-d vectors into a vector perpendicular to that area. It is mathematized by the definition

$$\left.\begin{array}{lr} \star : \Lambda^2(E^3) \longrightarrow E^3 & (\text{"genus"}) \\ e_\ell \wedge e_k \rightsquigarrow \star(e_\ell \wedge e_k) = e_m\, \epsilon^m{}_{\ell k} & (\text{"differentia"}) \end{array}\right\} \quad (30.7)$$

Apply this isomorphic map to the moment of density ("shear force per unit area") and obtain

$$\star(\overleftrightarrow{T}) = \star(d\mathcal{P}\wedge\overleftrightarrow{T})$$

$$= e_m \epsilon^m{}_{\ell k} F^k_{ij}\, dx^\ell\wedge dx^i\wedge dx^j/2! \equiv \underset{\sim}{\vec{T}} \qquad (\text{"\underline{torque} 3-form"})$$

This is the vector ("torque")-valued volume form.

The vectorial coefficient of this 3-form is readily obtained by noting that

$$dx^\ell\wedge dx^i\wedge dx^j = [\ell\, i\, j]\; dx^1\wedge dx^2\wedge dx^3.$$

Consequently, the torque 3-form is

$$\underset{\sim}{\vec{T}} = e_m g^{mn}\sqrt{g}\,[n\,\ell\,k]\,F^k_{ij}\,[\ell\, i\, j]\; dx^1\wedge dx^2\wedge dx^3/2!$$

Sum over the repeated indices of the product of the two permutation symbols. The result is the generalized Kronecker delta:

$$[n\,\ell\,k][\ell\, i\, j] = (-)\,\delta^{ij}_{nk} = (-)\begin{vmatrix}\delta^i_n & \delta^i_k\\ \delta^j_n & \delta^j_k\end{vmatrix} = -\left(\delta^i_n\delta^j_k - \delta^i_k\delta^j_n\right)$$

It follow that the vectorial 3-form is

$$\underset{\sim}{\vec{T}} = e_m\, g^{mn}(-)F^k{}_{kn}\sqrt{g}\; dx^1\wedge dx^2\wedge dx^3$$

or explicitly, using Eq. (30.3)

$$\underset{\sim}{\vec{T}} = e_m \epsilon^m{}_{\ell k}\, r^\ell E^k q N\sqrt{g}\; dx^1\wedge dx^2\wedge dx^3$$

$$= \frac{Nq}{g}\begin{vmatrix} e_1 & e_2 & e_3\\ r_1 & r_2 & r_3\\ E_1 & E_2 & E_3\end{vmatrix}\sqrt{g}\; dx^1\wedge dx^2\wedge dx^3$$

III. Moment of Curvature-induced Rotation.

The mathematical method of moments, which Cartan introced in terms of his differential forms, applies also to higher dimensional spaces, including spacetime, which is 4-dimensional. This application is mandatory if one wishes to understand gravitation on a level that approaches that of electromagnetism*

*\footnote{See Box 15.1.H – 15.1.I}

However, understanding the meaning of Einstein's tensor on the l.h.s. of Eq. (30.1) requires a different type of moment, namely the moment of curvature-induced rotation which is depicted in Figure 30.2 below. A vector parallel transported around the boundary of a face gets rotated by an angle such as the one depicted in the red parallelograms in Figure 30.2. In that figure this curvature-induced rotation for the $\vec{v}$-$\vec{t}$ spanned face is

$$\overleftrightarrow{\mathcal{R}} = e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{|\alpha\beta|}\, dx^\alpha \wedge dx^\beta (\vec{v}, \vec{t})$$

This rotation is a linear combination of bivectors, each one in the plane spanned by

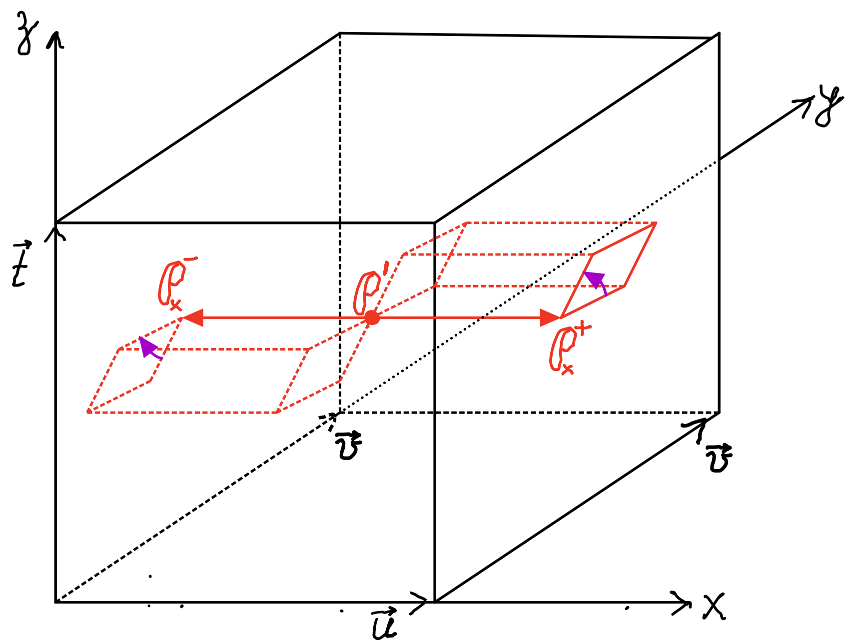


Figure 30.2 Moment of rotation induced the curvature permeating two opposing faces of a 3-cube.
P' = "fulcrum", an arbitrarily located point in or near the 3-cube.

the pair e_λ and e_μ; the angle of rotation is

$$\theta_{\lambda\mu} = R^{\lambda\mu}{}_{|\alpha\beta|}\, dx^\alpha \wedge dx^\beta(\vec{v}, \vec{t}).$$

There are two lever arms emanating from the arbitrarily located fulcrum point $\mathcal{P}'$;

$$\overrightarrow{\mathcal{P}^+ - \mathcal{P}'} \text{ and } (\mathcal{P}^- - \mathcal{P}'),$$

and terminating at $\mathcal{P}^+$ and $\mathcal{P}^-$ on two opposing faces of the 3-cube in Figure 30.2. However, their difference,

$$(\mathcal{P}^+ - \mathcal{P}') - (\mathcal{P}^- - \mathcal{P}') = \mathcal{P}^+ - \mathcal{P}^-$$

is independent of the fulcrum $\mathcal{P}'$. Instead, it is a lever arm that connects the two opposing.

IV. Cartan's Unit Tensor as a Fulcrum-Based Lever arm.

From the collection of lever arms ("displacement vectors") in a coordinate neighborhood, focus on those that emanate from a common fulcrum $\mathcal{P}'$.

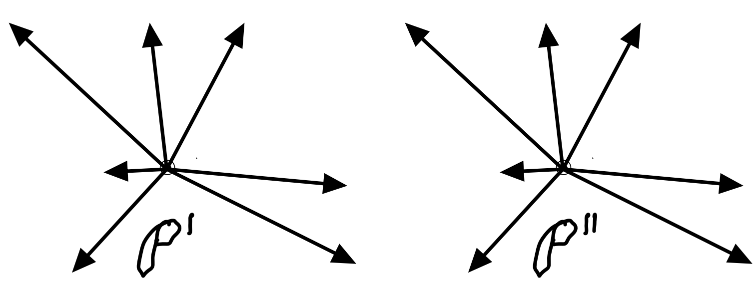


Figure 30.3 The domain of $d\mathcal{P}|_{\mathcal{P}'}$ is the vector space $V_{\mathcal{P}'}$, while that of $d\mathcal{P}|_{\mathcal{P}''}$ is the vector space $V_{\mathcal{P}''}$ at $\mathcal{P}''$.

These lever arms are mathematized quite trivially in terms of Cartan's unit tensor at fulrum $\mathcal{P}'$,

$$d\mathcal{P}\Big|_{\mathcal{P}'} : \qquad V_{\mathcal{P}'} \longrightarrow V_{\mathcal{P}'}$$

$$\mathcal{P}-\mathcal{P}' = \Delta x^\tau \frac{\partial}{\partial x^\tau} \rightsquigarrow d\mathcal{P}(\mathcal{P}-\mathcal{P}') = e_\sigma \langle dx^\sigma, \Delta x^\tau \frac{\partial}{\partial x^\tau} \rangle = e_\sigma \Delta x^\sigma = \mathcal{P}-\mathcal{P}'$$

One says that $d\mathcal{P}|_{\mathcal{P}'}$ is a vector at $\mathcal{P}'$ that refers to an as-yet-unspecified displacement away from $\mathcal{P}'$.

$d\mathcal{P}$ at fulcrum $\mathcal{P}'$ is to be contrasted with $d\mathcal{P}$ at a different fulcrum $\mathcal{P}''$.

V. Moment of Rotation as a Vector-valued volume form.