Lecture 30-Appendix

Holonomy: Rotation Generated by Curvature

(30A.1)

Holonomy is rotation induced by parallel transport around closed curves with a common base point.

A holonomy is mathematized by a linear transformation which depends on the curvature integrated over the area bounded by a curve, say C. This linear tranformation is given by

$$T^{\alpha}{}_{\beta}[C] = \exp \oint\!\!\oint_{S} R^{\alpha}{}_{\beta u v}(\lambda,\eta)\, d\lambda\, d\eta$$
$$= \exp \oint\!\!\oint_{S} R^{\alpha}{}_{\beta u v}\left(x^{\sigma}(\lambda,\eta)\right) d\lambda\, d\eta, \qquad (30A.1)$$

where S is the area spanned by $x^{\sigma}(\lambda,\eta)$ and

$$u^{\rho} = \frac{\partial x^{\rho}(\lambda,\eta)}{\partial \lambda}$$
$$v^{\sigma} = \frac{\partial x^{\sigma}(\lambda,\eta)}{\partial \eta}$$

are the component of the vectors $\frac{\partial}{\partial \lambda}$ and $\frac{\partial}{\partial \eta}$ tangent to S.

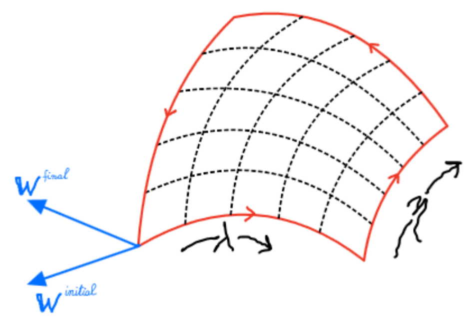


Figure 30A.1 When a vector $W^{initial}$ is parallel transported around the boundary of the 2-d surface

$$S = \{P(\lambda,\eta):\ 0 \le \lambda, \eta \le 1\},$$

the specific nature of this parallel tranport (as mathematized by its curvature tensor) transforms this vector into W^{final}.

30A.2

Where does the transformation Eq. (30A.1) come from, and how does one apply it?

1. Let $P(\lambda,\eta)$: $x^\alpha(\lambda,\eta)$ be a 2-d surface S spanned by λ and η.

2. Let

$$\mathbf{u} = \frac{\partial}{\partial\lambda}\Big)_\eta \quad \text{and} \quad \mathbf{v} = \frac{\partial}{\partial\eta}\Big)_\lambda$$

be the coordinate tangents to S.

3. Partition S into a union of small rectangles, each one being spanned by two small vectors

$$\Delta \mathbf{u}_{ij} = \Delta\lambda \overbrace{\frac{\partial x^\alpha(\lambda,\eta)}{\partial\lambda}}^{u^\alpha_{ij}}\Bigg|_{(\lambda_i,\eta_j)} \overbrace{\frac{\partial}{\partial x^\alpha}}^{e_\alpha} \equiv \Delta\lambda\, u^\alpha_{ij}\, e_\alpha$$

and

$$\Delta \mathbf{v}_{ij} = \Delta\eta \underbrace{\frac{\partial x^\beta(\lambda,\eta)}{\partial\eta}}_{v^\beta_{ij}}\Bigg|_{(\lambda_i,\eta_j)} \underbrace{\frac{\partial}{\partial x^\beta}}_{e_\beta} \equiv \Delta\lambda\, v^\beta_{ij}\, e_\beta$$

Thus the $(i,j)^{th}$ rectangle is spanned by

$$(\Delta \mathbf{u}_{ij}, \Delta \mathbf{v}_{ij})$$

4. Let $P(\lambda_i,\eta_j) \equiv P_0(i,j)$ be a reference point on the boundary of the $(i,j)^{th}$ rectangle.

5. Let $\mathbf{w} = w^\gamma(P)\frac{\partial}{\partial x^\gamma} \equiv w^\gamma(P)\, e_\gamma(P)$ be a vector field on S, and let

global $$\mathbf{w}(P(\lambda_i,\eta_j)) \equiv \mathbf{w}_{ij}(P_0) = e_\gamma\, w^\gamma_{ij}$$

be the vector value of $\mathbf{w}$ at $P_0(i,j)$

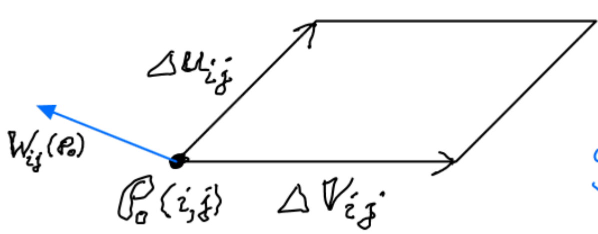


Figure 30A.2

30A.3

6. Let $W_{ij}^{circuit}(P_0)$ be the result of parallel transporting $W_{ij}(P_0)$ around the perimeter of the $(i,j)^{th}$ rectangle:

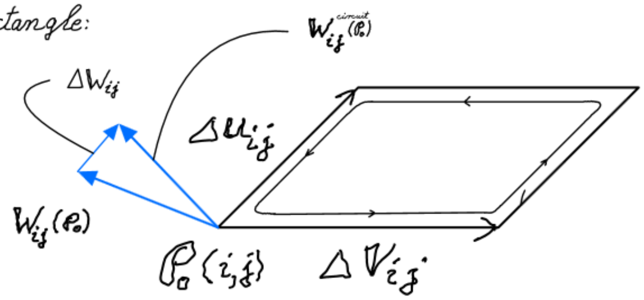


Figure 30A.3

Let ΔW_{ij} be the vectorial difference between the parallel transported vector and the one preexisting at $P_0(i,j)$:

$$\Delta W_{ij} \equiv W_{ij}^{circuit}(P_0) - W_{ij}(P_0) = e_\alpha \underset{\sim}{R}{}^{\alpha}{}_{\beta}\Big|_{P_0(i,j)} (\Delta u_{ij}, \Delta V_{ij})\, w^{\beta}_{ij}$$

$$\equiv e_\alpha \left(d\omega^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\gamma} \wedge \omega^{\gamma}{}_{\beta}\right)\Big|_{P_0(i,j)} (\Delta u_{ij}, \Delta V_{ij})\, w^{\beta}_{ij}$$

$$= e_\alpha \underbrace{R^{\alpha}{}_{\beta\rho\sigma}\Big|_{P_0(i,j)} u^{\rho}_{ij}\, v^{\sigma}_{ij}}_{\begin{array}{c}|||\\ R^{\alpha}{}_{\beta uv}(i,j)\end{array}}\, w^{\beta}_{ij}\, \Delta\lambda\, \Delta\eta$$

This difference is the change in the vector $w_{ij}(P_0)$ due to circumnavigational parallel transport around the boundary of S.

It follows that this circumnavigation implies

$$W_{ij}^{circuit}(P_0) = e_\alpha(i,j)\, T^{\alpha}{}_{\beta}(i,j)\, w^{\beta}_{ij}.$$

Here

$$T^{\alpha}{}_{\beta}(i,j) = \delta^{\alpha}_{\beta} + R^{\alpha}{}_{\beta uv}(i,j)\,\Delta\lambda\,\Delta\eta$$

is a parallel transport induced linear transformation at $P_0(i,j)$. It differs infinitesimally from the identity transformation.

7. Let P_0^{glob} be a single global reference point on or off the boundary of S.

30.A.4

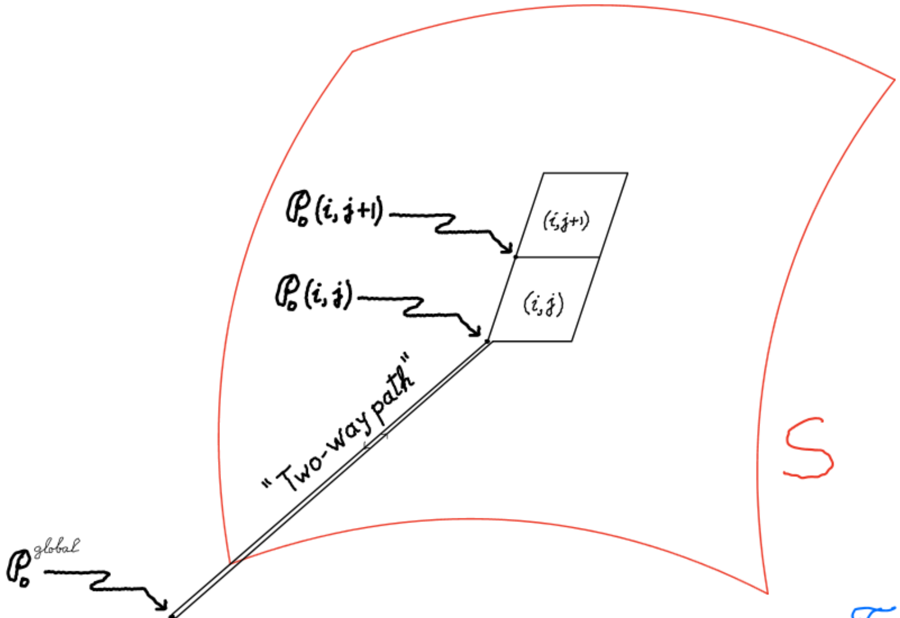


Figure 30A.4

Let (i,j) and $(i,j+1)$ refer to two adjacent area elements of S.

Consider a two-way path

$$\mathcal{P}_0^{global} \longleftrightarrow \mathcal{P}_0(i,j)$$

which connects that single reference point with the point $\mathcal{P}_0(i,j)$ at area element (i,j).

8.a) Starting with some vector at $\mathcal{P}_0^{global}$, say $w_{ij}(\mathcal{P}_0^{global})$, parallel transporting it to $\mathcal{P}_0(i,j)$, carrying it around the $(i,j)^{th}$ area element, and then back to $\mathcal{P}_0^{global}$, one obtains

$$W_{ij}^{circuit}(\mathcal{P}_0^{global}) = e_\alpha(\mathcal{P}_0^{global})\, T^\alpha{}_\beta(i,j)\, w_{ij}^\beta(\mathcal{P}_0^{global}). \qquad (30A.2)$$

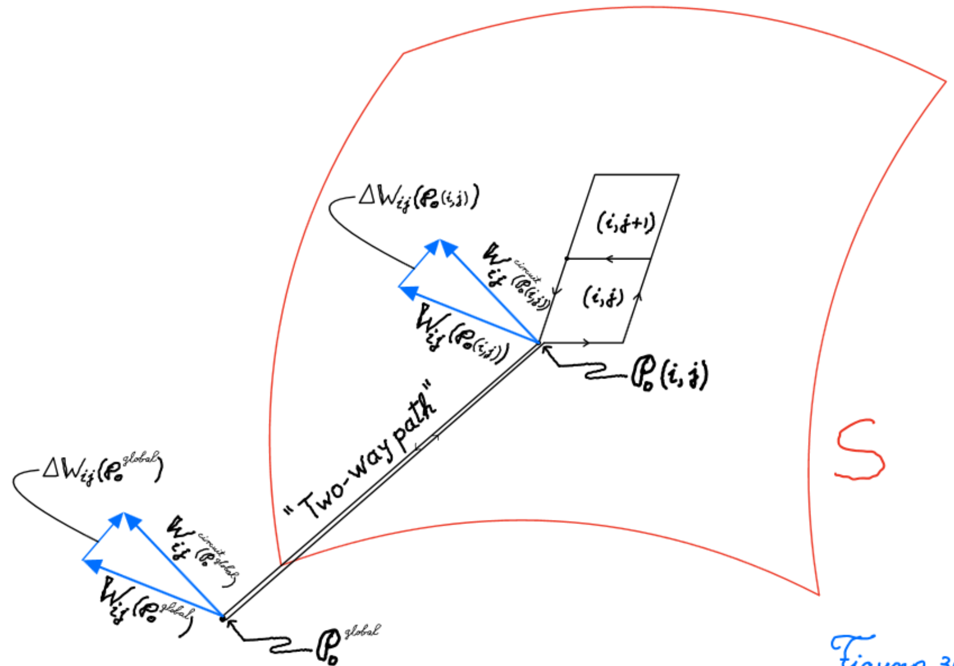


Figure 30A.5

8.b) Next repeat the same process, but circumnavigate the $(i,j+1)^{th}$ element of area instead.

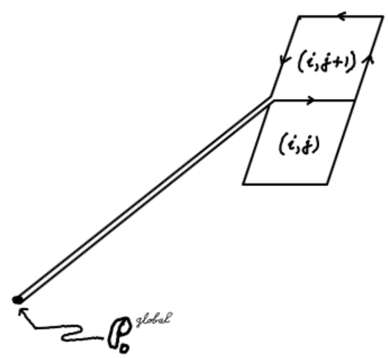


Figure 30A.6

One obtains following Eq.(30A.2) on page 30A.4

$$W^{circuit}_{ij+1}(\mathcal{P}_0^{global}) = e_\alpha(\mathcal{P}_0^{global})\ T^\alpha{}_\gamma(i,j+1)\ w^\gamma_{ij+1}(\mathcal{P}_0^{global}),$$

or in terms of coordinate components

$$\left(w^{circuit}_{i,j+1}\right)^{\alpha} = T^{\alpha}{}_{\beta}(i,j+1)\, w^{\beta}_{i,j+1}$$ (30A.6)

$$= T^{\alpha}{}_{\gamma}(i,j+1)\, T^{\gamma}{}_{\beta}(i,j)\, w^{\beta}_{ij}$$

9. Notice that this result is equivalent to starting with $w_{ij}(P_0^{global})$ and cirmumnavigating both area elements (i,j) and $(i,j+1)$ at once. This is because the contributions to $w^{circuit}_{i,j+1}$ from adjecent sides concel out.

10. In order to extend the process to all area elements, it is more economical to use matrix notation

$$\left[w^{circum}_{i,j+1}\right] = T(i,j+1)\, T(i,j)\left[w_{ij}\right].$$

Applied to all area elements of S, one obtains

$$\left[w^{circum}\right] = \prod_i \prod_j T(i,j)\left[w\right]$$

$$= \prod_i \prod_j \left[I + R(i,j)\Delta\lambda\Delta\eta\right]\left[w\right]$$

Here

$$\left[I + R(i,j)\Delta\lambda\Delta\eta\right]^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} + R^{\alpha}{}_{\beta u v}(i,j)\,\Delta\lambda\Delta\eta\,.$$

11. The product of the line integrals

$$\prod_i \prod_j \left[I + R(i,j)\Delta\lambda\Delta\eta\right] \qquad (30A.3)$$

can be converted into an area integral over the two-dimensional area S. This is achieved by letting the partitioning of S into area elements become finer so that $\Delta u_{ij}, \Delta v_{ij}$ for the associated rectangles tend to zero.

To achieve such a conversion, take the limit of the logarithm of the matrix product Eq. (30A.3): (30A.7)

$$\lim_{\substack{\lambda_{i+1}\to\lambda_i \\ \eta_{j+1}\to\eta_j}} \log \prod_i \prod_j \left[I + \mathcal{R}(i,j)\Delta\lambda\Delta\eta \right] \qquad \begin{array}{l}\Delta\lambda = \lambda_{i+1}-\lambda_i \\ \Delta\eta = \eta_{j+1}-\eta_j\end{array}$$

$$= \lim_{\substack{\lambda_{i+1}\to\lambda_i \\ \eta_{j+1}\to\eta_j}} \sum_i \sum_j \log \left[I + \mathcal{R}(i,j)\Delta\lambda\Delta\eta \right]$$

$$= \lim_{\substack{\lambda_{i+1}\to\lambda_i \\ \eta_{j+1}\to\eta_j}} \sum_i \sum_j \mathcal{R}(i,j)\Delta\lambda\Delta\eta$$

It follows that the in-the-limit infinite product is

$$\boxed{\lim_{\substack{\lambda_{i+1}\to\lambda_i \\ \eta_{j+1}\to\eta_j}} \prod_i \prod_j \left[I + \mathcal{R}(i,j)\Delta\lambda\Delta\eta \right] = \exp \oint\!\!\oint_S \mathcal{R}(\lambda,\eta)\, d\lambda\, d\eta = \left[\exp \oint\!\!\oint_S R^{\alpha}{}_{\beta\mu\nu}(\lambda,\eta)\, d\lambda\, d\eta\right]}$$

and that the rotation experienced by a vector parallel transported around a loop that encloses the surface S is the linear tranformation

$$\boxed{\mathbb{W} = e_\sigma w^\sigma \xrightarrow{\exp \oint\!\!\oint_S \mathcal{R}(\lambda,\eta)\, d\lambda\, d\eta} \mathbb{W}^{circum} = e_\alpha \exp \oint\!\!\oint_S R^{\alpha}{}_{\beta\mu\nu}(\lambda,\eta)\, d\lambda\, d\eta\; w^\beta}$$

CONCLUSION I

By starting with some vector $\mathbb{W}$ at some point event on a closed curve C, which is the boundary enclosing a surface S, one finds, upon parallel transporting $\mathbb{W}$ around C back to its starting point event, that

$$\mathbb{W} \rightsquigarrow \mathbb{W}^{circum}$$

is a linear transformation. It is mathematized by the surface (30A.8)
integral of the curvature. The basis representation of this
C-determined linear transformation is

$$T^{\alpha}{}_{\beta}[C] = \exp \oiint_{S} R^{\alpha}{}_{\beta u v}(\lambda,\eta)\, d\lambda\, d\eta \qquad (30A.4)$$

$$\equiv \exp \oiint_{S} R^{\alpha}{}_{\beta\rho\sigma}(x^{\gamma}(\lambda,\eta))\, u^{\rho}(\lambda,\eta)\, v^{\sigma}(\lambda,\eta)\, d\lambda\, d\eta$$

For metric compatible parallel tranport

$$T^{\alpha\beta}[C] = \exp \oiint_{S} R^{\alpha\beta}{}_{uv}(\lambda,\eta)\, d\lambda\, d\eta$$

is a rotation matrix. Thus, for every curve C there exists a rotation transformation. This transformation is one which is orthonormal and the anti-symmetric matrix $\oiint R^{\alpha\beta}{}_{uv}(\lambda,\eta)\, d\lambda\, d\eta$ is its gerator.

CONCLUSION II

It is easy to see that two curves can (with the same starting point) be strung together to form a third curve

$$C_3 = C_2 \circ C_1 .$$

The corresponding rotation transformation is

$$T[C_3] = T[C_2] \circ T[C_1].$$

Thus, metric-compatible parallel transport establishes a mapping between closed curves and the group of Euclidean/Lorentz rotations.

CONCLUSION III

The agreement between rotation induced by (a) parallel transport around closed <u>curves</u> with a fixed base point and (b) by the exponentiation of the corresponding <u>area</u> integrals of curvature form implies that the

30A.9

physical basis of the concept "curvature" is necessitated by the observation that things get rotated and these rotation form a group — the Holonomy Group at a base point and whose elements are generated by the curvature field permeating the area of the loops.

Application to a Two-sphere

As an application consider the holonomy of a meridean circle on a two sphere of radius a, whose is

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta \, d\varphi^2$$

and whose curvature relative to the coordinate bases (30A.10)

$$\left\{\frac{\partial}{\partial\theta},\frac{\partial}{\partial\varphi}\right\} \quad \& \quad \{d\theta, d\varphi\}$$

has components

$$R^{\theta}{}_{\varphi\theta\varphi}=\sin^2\theta$$

$$R^{\theta}{}_{\theta\theta\varphi}=0$$

$$R^{\varphi}{}_{\theta\theta\varphi}=g^{\varphi\varphi}R_{\varphi\theta\theta\varphi}=-g^{\varphi\varphi}g_{\theta\theta}R^{\theta}{}_{\varphi\theta\varphi}=-1$$

$$R^{\varphi}{}_{\varphi\varphi\varphi}=0.$$

Its components relative to the orthonormal basis

$$\left\{\frac{1}{a}\frac{\partial}{\partial\theta},\frac{1}{a\sin\theta}\frac{\partial}{\partial\varphi}\right\} \quad \& \quad \{a\,d\theta,\ a\sin\theta\,d\varphi\}$$

are

$$\hat{R}^{\theta}{}_{\varphi\theta\varphi}=(g_{\theta\theta}g^{\varphi\varphi}g^{\theta\theta}g^{\varphi\varphi})^{1/2}R^{\theta}{}_{\varphi\theta\varphi}=\frac{1}{a^2}$$

$$\hat{R}^{\theta}{}_{\theta\theta\varphi}=0$$

$$\hat{R}^{\varphi}{}_{\theta\theta\varphi}=\hat{R}_{\varphi\theta\theta\varphi}=-\hat{R}_{\theta\varphi\theta\varphi}=-\hat{R}^{\theta}{}_{\varphi\theta\varphi}=\frac{-1}{a^2}$$

$$\hat{R}^{\varphi}{}_{\varphi\theta\varphi}=0$$

In order to calculate the rotation, Eq. (30A.4) page 30A.8, first evaluate the integral

$$\oiint_S \hat{R}^{k}{}_{\ell mn}\,\hat{\omega}^m\wedge\hat{\omega}^n.$$

Doing so for the domain $S=\{\theta,\varphi:\ 0\leq\theta\leq\theta_0,\ 0\leq\varphi<2\pi\}$, which is the northern polar cap bounded by the meridean $\theta=\theta_0$, one finds

$$\oiint \hat{R}^{k}{}_{\ell|mn|}\,\omega^m\wedge\omega^n=\int_0^{2\pi}\!\!\int_0^{\theta_0}\underbrace{\hat{R}^{k}{}_{\ell\theta\varphi}}_{\begin{bmatrix}0&1\\-1&0\end{bmatrix}\frac{1}{a^2}}\,a\,d\theta\ a\sin\theta\,d\varphi$$

30A.11

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} 2\pi(1-\cos\theta_0) \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \alpha \equiv K\alpha \rightarrow \quad K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\alpha = 2\pi(1-\cos\theta_0)$$

Second, the rotation, Eq. (30A.4) on page 30A.8, is therefore

$$\exp K\alpha = \sum_{n=0}^{\infty} K^n \frac{\alpha^n}{n!}.$$

Using $K^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$

$K^3 = -K$

$K^4 = I,$

one obtains

$$\exp K\alpha = I\left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \cdots\right) + K\left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \cdots\right)$$

$$= I\cos\alpha + K\sin\alpha$$

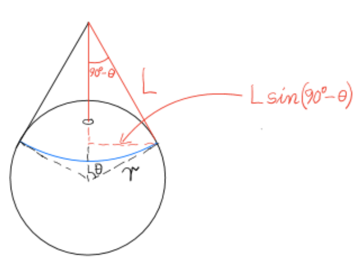
$$= \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

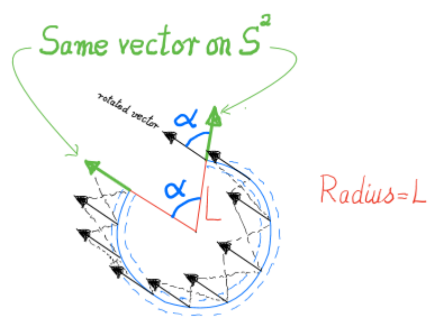
Thus one has

$$\exp \oiint \hat{R}^k{}_{\ell uv}(\lambda,\eta)\, d\lambda\, d\eta = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

where, as expected, $\alpha = 2\pi(1-\cos\theta_0)$ is the angular rotation experienced by any vector parallel transported around the meridean $\theta = \theta_0$.

This experienced rotation can also be obtained from the following pictorial argument





30A.12

This angular rotation α due to parallel transport can also be established analytically by solving the equations for parallel transport

$$\nabla_{v} w = 0$$

which for $v = \dot{x}^{\beta} \frac{\partial}{\partial x^{\beta}}$ is equivalent to

$$\dot{w}^{\alpha} + \Gamma^{\alpha}{}_{\beta\gamma} \dot{x}^{\beta} w^{\gamma} = 0$$