Lecture 31 & 32

The 2plus2 Decomposition
of
Spacetime

I Spherically Symmetric Tensor Fields

A tensor field is said to be symmetric if it is invariant under the transformation generated by the vector field

$\xi^\mu(x^\alpha)\dfrac{\partial}{\partial x^\mu}$,

$$\begin{aligned} x^\alpha &\longrightarrow x'^\alpha = x^\alpha + \varepsilon\,\xi^\alpha(x^\delta) \\ x'^\delta &\longrightarrow x^\delta = x'^\delta - \varepsilon\,\xi^\delta(x'^\alpha)\,, \quad \varepsilon \ll 1. \end{aligned} \tag{31.1}$$

Thus, whenever

$$v_\mu(x^\alpha)\, d(x^\mu + \varepsilon\,\xi^\mu(x^\delta)) = v_\mu(x^\alpha)\, dx^\alpha$$

$$g_{\mu\nu}(x^\alpha + \varepsilon\,\xi^\alpha)\, d(x^\mu + \varepsilon\,\xi^\mu(x^\delta))\, d(x^\nu + \varepsilon\,\xi^\nu(x^\delta)) = g_{\mu\nu}(x^\alpha)\, dx^\mu dx^\nu$$

$$v^\mu(x'^\delta)\frac{\partial}{\partial x'^\mu} \equiv v^\mu(x^\delta + \varepsilon\,\xi^\delta)\frac{\partial x^\delta}{\partial x'^\mu}\frac{\partial}{\partial x^\delta} = v^\mu\frac{\partial}{\partial x^\mu}$$

to first order in ε, one says that the covector field $v_\mu\, dx^\mu$, the tensor field $g_{\mu\nu}\, dx^\mu dx^\nu$, and the vector field $v^\mu \dfrac{\partial}{\partial x^\mu}$ are invariant under the ξ^μ-generated transformation.

From their representations relative to the coordinate system

$$\{x^\mu : \underbrace{x^0,\ x^1}_{\substack{\text{longitudinal}\\ \text{coordinates}\\ \{x^A:\ A=0,1\}}},\ \underbrace{x^2=\theta,\ x^3=\varphi}_{\substack{\text{transverse}\\ \text{coordinates}\\ \{x^a:\ a=\theta,\varphi\}}}\}$$

one infers that the

$$\xi^\mu_\theta \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \theta} \quad \text{and} \quad \xi^\mu_\varphi \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \varphi}$$

generated transformations are symmetry transformations. This is because they leave each of the following geometrical and physical tensor fields in spacetime invariant.

1. Metric tensor:

$$g_{\mu\nu}\,dx^\mu dx^\nu = g_{AB}(x^C)\,dx^A dx^B + \overbrace{r^2(x^C)\underbrace{(d\theta^2 + \sin^2\theta\, d\varphi^2)}_{\gamma_{ab}\,dx^a dx^b}}^{g_{ab}\,dx^a dx^b} : \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{g_{AB}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{r^2\gamma_{ab}} \\ 0 & 0 & & \end{array}\right]$$

$g_{AB}(x^C)\,dx^A dx^B$

$r^2(x^C)$

$\gamma_{ab}\,dx^a dx^b = d\theta^2 + \sin^2\theta\, d\varphi^2$

2. Momenergy tensor:

$$t_{\mu\nu}\,dx^\mu dx^\nu = t_{AB}(x^a)\,dx^A dx^B + \overbrace{\underbrace{t(x^a)\,r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)}_{t\,g_{ab}\,dx^a dx^b}}^{t_{ab}\,dx^a dx^b} : \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{t_{AB}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{t\,g_{ab}} \\ 0 & 0 & & \end{array}\right]$$

$t_{AB}(x^a)\,dx^A dx^B$

$t(x^a)$

3. General covector:

$$v_\mu\,dx^\mu : \begin{bmatrix} v_A \\ 0 \\ 0 \end{bmatrix}$$

4. The Klein-Gordon field equation (31.3)

$$\Box\psi - \frac{m^2c^2}{\hbar^2}\psi \equiv \left(g^{\mu\nu}\frac{\partial\psi}{\partial x^\mu}\right)_{;\nu} - \lambda_c^{-2}\psi$$

$$= \left(g^{AB}\frac{\partial\psi}{\partial x^A}\right)_{|B} + \frac{1}{r^2}\left(\gamma^{ab}\frac{\partial\psi}{\partial x^a}\right)_{:b} - \frac{1}{\lambda^2}\psi$$

$$= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^A}\left[\sqrt{-g}\, g^{AB}\frac{\partial\psi}{\partial x^B}\right] + \frac{1}{r^2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial^2\psi}{\partial\varphi^2}\right\} - \frac{1}{\lambda_c^2}\psi$$

Let $\psi = \psi_{\ell m}(x^a)\, Y_\ell^m(\theta,\varphi)$ be a spherical normal mode solution. It satisfies

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial^2\psi}{\partial\varphi^2} = -\ell(\ell+1)\psi$$

The amplitude $\psi_{\ell m}(x^a)$ satifies

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^A}\left[\sqrt{-g}\, g^{AB}\frac{\partial\psi_{\ell m}}{\partial x^B}\right] - \left(\frac{\ell(\ell+1)}{r^2} + \frac{1}{\lambda_c^2}\right)\psi_{\ell m} = 0$$

4. Einstein tensor

$$G_{\mu\nu}\,dx^\mu dx^\nu \equiv \left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R\right) dx^\mu dx^\nu \qquad \left[\begin{array}{c:c} G_{AB} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \hdashline \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & G_{ab} \end{array}\right]$$

$$= G_{AB}\,dx^A dx^B + \underbrace{G_{ab}}_{\frac{1}{2} G_d{}^d g_{ab}}\,dx^a dx^b$$

$$G_{AB} = \frac{1}{r^2}\left\{-2\, r_{;A|B} + \left(2 r\, r_{;c}{}^{|c} + r_{;c}\, r^{;c} - 1\right) g_{AB}\right\}$$

$$G_{ab} \equiv \frac{1}{2} G_d{}^d g_{ab} = \left(\frac{r_{;c}\, r^{;c}}{r} - \mathcal{R}\right) g_{ab}$$

where $\mathcal{R}$ is the Gaussian curvature defined by

$$^{(2)}R^{AB}{}_{CD} = \mathcal{R}\left(\delta^A_C \delta^B_D - \delta^A_D \delta^B_C\right)$$

5. The Einstein field equations

$$G_{AB} = -2 r\, r_{;A|B} + g_{AB}\left(2 r\, r_{;c}{}^{|c} + r_{;c}\, r_{;D}\, g^{CD} - 1\right) = \frac{8\pi G}{c^2} r^2\, t_{AB}$$

$$\frac{1}{2} G_a{}^a = \frac{r_{;C|D}\, g^{CD}}{r} - \mathcal{R} = \frac{8\pi G}{c^2} t$$

6. The conservation equation

$$G_\mu{}^\nu{}_{;\nu} \equiv 0 = t_\mu{}^\nu{}_{;\nu}$$

$$\left(r^2 G_A{}^B\right)_{|B} - r\, r_{;A}\, G_a{}^a = 0$$

$$\left(r^2 t_A{}^B\right)_{|B} - r\, r_{;A}\, 2t = 0$$