Lecture 31 & 32

The 2plus2 Decomposition
of
Spacetime

I Spherically Symmetric Tensor Fields

A tensor field is said to be symmetric if it is invariant under the transformation generated by the vector field

$\xi^\mu(x^\alpha)\frac{\partial}{\partial x^\mu}$,

$$x^\alpha \longrightarrow x'^\alpha = x^\alpha + \varepsilon\, \xi^\alpha(x^\gamma) \qquad (31.1a)$$

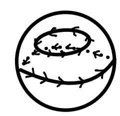
$$x'^\gamma \longrightarrow x^\gamma = x'^\gamma - \varepsilon\, \xi^\gamma(x'^\alpha)\,, \qquad \varepsilon \ll 1. \qquad (31.1b)$$

Thus, whenever

a) $v_\mu(x^\alpha+\varepsilon\xi^\alpha(x^\gamma))\, d(x^\mu+\varepsilon\xi^\mu(x^\gamma)) \qquad = v_\mu(x^\alpha)\,dx^\alpha$

b) $g_{\mu\nu}(x^\alpha+\varepsilon\xi^\alpha)\, d(x^\mu+\varepsilon\xi^\mu(x^\gamma))\, d(x^\nu+\varepsilon\xi^\nu(x^\gamma)) \quad = g_{\mu\nu}(x^\alpha)\,dx^\mu dx^\nu$

c) $v^\mu(x'^\gamma)\frac{\partial}{\partial x'^\mu} \equiv v^\mu(x^\gamma+\varepsilon\xi^\gamma(x^\beta))\,\frac{\partial(x'^\delta-\varepsilon\xi^\delta(x'^\alpha))}{\partial x'^\mu}\frac{\partial}{\partial x^\delta} = v^\mu(x^\delta)\frac{\partial}{\partial x^\mu}$



to first order in ε, one says that the covector field $v_\mu\, dx^\mu$, the tensor field $g_{\mu\nu}\,dx^\mu dx^\nu$, and the vector field $v^\mu\frac{\partial}{\partial x^\mu}$ are invariant under the ξ^μ-generated transformation.*

From their representations relative to the coordinate system

$$\{x^\mu : \underbrace{x^0,\ x^1}_{\substack{\text{longitudinal}\\ \text{coordinates}\\ \{x^A:\ A=0,1\}}},\ \underbrace{x^2=\theta,\ x^3=\varphi}_{\substack{\text{transverse}\\ \text{coordinates}\\ \{x^a:\ a=\theta,\varphi\}}}\}$$

one infers that the

* \footnote{ A noteworthy feature of the invariance conditions a)-c) is that they do not depend on any law of parallel transport. Instead they depend on the differential structure of the manifold. Nevertheless, they do obey that law and its rules of differentiation when its covariant derivative is introduced.

Of the highlighted examples a)-c), invariance condition b) is the most important. This is because it yields a tensorial differential equation. Any non-zero solution must satisfy it in order that there exist a symmetry point transformation that is generated by its vectorial generator.

The invariance condition

$$g_{\mu\nu}(x^\alpha+\varepsilon\xi^\alpha)\,d(x^\mu+\varepsilon\xi^\mu(x^\delta))\,d(x^\nu+\varepsilon\xi^\nu(x^\delta))=g_{\mu\nu}(x^\alpha)\,dx^\mu dx^\nu$$

implies

$$g_{\mu\nu,\alpha}\,\xi^\alpha+g_{\alpha\nu}\,\xi^\alpha{}_{,\mu}+g_{\mu\alpha}\,\xi^\alpha{}_{,\nu}=0$$

This is obviously the same as

$$g_{\mu\nu;\alpha}\,\xi^\alpha+g_{\alpha\nu}\,\xi^\alpha{}_{;\mu}+g_{\mu\alpha}\,\xi^\alpha{}_{;\nu}=0. \qquad (31.2)$$

But a metric compatible parallel transport implies

$$g_{\mu\nu;\alpha}=0.$$

Furthermore, the product rule for covariant differentiating implies

$$(g_{\alpha\nu}\,\xi^\alpha)_{;\mu}=g_{\alpha\nu;\mu}\,\xi^\alpha+g_{\alpha\nu}\,\xi^\alpha{}_{;\mu}.$$

Consequently, the symmetry condition Eq. (31.2) reduces to

$$\xi_{\nu;\mu}+\xi_{\mu;\nu}=0$$

The importance of the invariance in example c) derives from the fact that it pertains to vector fields as compared to covector fields of example a). That invariance condition requires an extra "push forward" step for its mathematization.

The line of reasoning and its context are as follows, but first recall how one arrives at the concept of a vector. Consider a pair of points

$$\mathcal{P}_1 = \{x_1^\mu\} \text{ and } \mathcal{P}_2 = \{x_2^\mu\}$$

separated by an infinitesimal "amount." This "amount" is mathematized by the partial derivatives of some but any differentiable scalar field $\psi(x^\alpha)$ in the neighborhood containing $\mathcal{P}_1$ and $\mathcal{P}_2$. That "amount" is given by the principal linear part

$$\vec{v}(\psi) = \Delta x^\mu \frac{\partial \psi}{\partial x^\mu}$$

of the increment

$$\psi(\mathcal{P}_2) - \psi(\mathcal{P}_1) = \underbrace{(x_2^\mu - x_1^\mu)}_{\Delta x^\mu} \frac{\partial \psi}{\partial x^\mu} + \cdots$$

as one moves from $\mathcal{P}_1$ to $\mathcal{P}_2$. Next omit explicit reference to any particular scalar ψ, but do so with the understanding that some scalar must always be used to obtain the number $\vec{v}(\psi)$.

One thereby arrives at the concept of a vector symbolized by

$$\vec{v} = \Delta x^\mu \frac{\partial}{\partial x^\mu} .$$

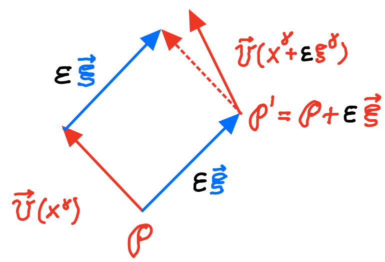


Figure 31.1

Apply the "push forward" step to the vector

$$\vec{v}(\mathcal{P}) = v^\mu(x^\gamma)\frac{\partial}{\partial x^\mu}$$

of the preexisting vector field whose value at the points $\mathcal{P}$ and $\mathcal{P}' = \mathcal{P} + \varepsilon\vec{\xi}$ are

$$\vec{v}(\mathcal{P}) \equiv v^\mu(x^\gamma)\frac{\partial}{\partial x^\mu} \quad \text{and} \quad \vec{v}(\mathcal{P} + \varepsilon\vec{\xi}) \equiv v^\mu(x^\gamma + \varepsilon\xi^\gamma)\frac{\partial}{\partial x'^\mu}\bigg|_{\mathcal{P}'} .$$

Pushing the $\vec{v}(\mathcal{P})$ forward from $\mathcal{P}$ to $\mathcal{P}' = \mathcal{P} + \varepsilon\vec{\xi}$ means to create a new vector, namely

$$v^\mu(x^\gamma)\frac{\partial}{\partial x'^\mu} = v^\mu(x^\gamma)\frac{\partial}{\quad}$$

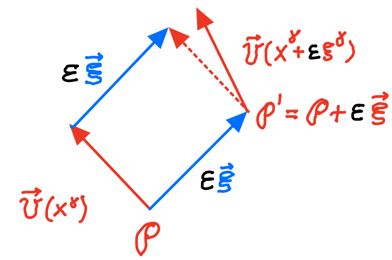
$$v^\mu(x'^\gamma)\frac{\partial}{\partial x'^\mu} \equiv v^\mu(x^\gamma+\varepsilon\xi^\gamma)\frac{\partial(x'^\delta-\varepsilon\xi^\delta(x'^\alpha))}{\partial x'^\mu}\frac{\partial}{\partial x^\delta} = v^\mu(x^\gamma)\frac{\partial}{\partial x^\mu},$$

implies

$$0 = (v^\mu{}_{,\alpha}\,\xi^\alpha - v^\gamma\,\xi^\mu{}_{,\gamma})\frac{\partial}{\partial x^\mu}$$

$$= [\vec{\xi}, \vec{v}]$$

$$= (\xi^\alpha\, v^\mu{}_{;\alpha} - v^\alpha \xi^\mu{}_{;\alpha})\frac{\partial}{\partial x^\mu}$$



$$\frac{\partial(x'^\mu+\varepsilon\xi^\mu(x'^\beta))}{\partial x'^\beta}\cdot\frac{\partial}{\partial x'^\beta} = \frac{\partial x^\mu}{\partial x'^\beta}\frac{\partial}{\partial x^\mu} \qquad v^\mu(x^\gamma)\frac{\partial}{\partial x^\mu}$$

$$v^\mu(x^\gamma)\frac{\partial x'^\beta}{\partial x^\mu}\frac{\partial}{\partial x'^\beta} \qquad v^\mu(x^\gamma+\varepsilon\xi^\gamma)\frac{\partial}{\partial x'^\gamma}$$

$$v^\mu(x^\gamma)\frac{\partial(x^\beta+\varepsilon\xi^\beta)}{\partial x^\mu}\frac{\partial}{\partial x'^\beta}$$

$$\frac{\partial}{\partial x'^\gamma}\psi(x'^\gamma) = \frac{\partial\,\psi(x^\gamma+\varepsilon\xi^\gamma)}{\partial x^\beta}\frac{\partial x^\beta}{\partial x'^\gamma} \qquad v^\mu(x^\gamma+\varepsilon\xi^\gamma)\frac{\partial}{\partial x^\mu} = v^\mu(x^\gamma)\frac{\partial}{\partial x^\alpha}$$

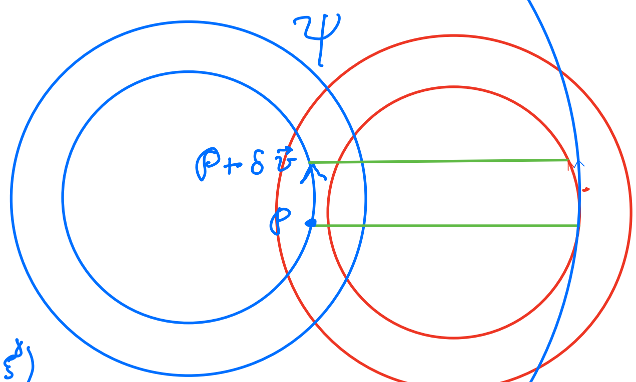
$$= \frac{\partial\,\psi(x^\gamma+\varepsilon\xi^\gamma)}{\partial x^\beta}\frac{\partial(x'^\beta+\varepsilon\xi^\beta(x'^\gamma))}{\partial x'^\gamma}$$

$$= \frac{\partial\psi}{\partial x^\beta}(\delta^\beta_\gamma + \varepsilon\,\xi^\beta{}_{,\gamma}) = \frac{\partial\psi}{\partial x^\gamma} + \frac{\partial\psi}{\partial x^\beta}\varepsilon\xi^\beta{}_{,\gamma}$$

$$v^\mu(x^\gamma)\frac{\partial\psi(x^\gamma)}{\partial x^\mu} = v^\mu(x^\gamma+\varepsilon\xi^\gamma)\frac{\partial\psi}{\partial y^\mu}(y^\gamma)\Big|_{y=x^\gamma+\varepsilon\xi^\gamma} = v^\mu(x^\gamma+\varepsilon\xi^\gamma)$$

$$f^\mu(x^\gamma+\varepsilon\xi^\gamma)$$

$$\psi(P) = \phi(P+\varepsilon\vec{\xi})$$



$y = T\,x$

$T_*\,\psi(x) = \psi(T^{-1}(x))$

$T_*\,\psi(T(x)) = \psi(x)$

$$\psi(x^\gamma+\varepsilon\xi^\gamma)$$
$$=$$

$$\phi(x^\gamma+\varepsilon\xi^\mu+\lambda v^\mu) = \psi(x^\gamma+\lambda v^\gamma)$$

$$\psi(x^\gamma) = \phi(x^\gamma+\varepsilon\xi^\gamma)$$

$$\phi(x^\gamma) + \phi_{,\gamma}\,\varepsilon\xi^\mu + \phi_{,\gamma}\lambda v^\mu = \psi(x^\gamma) + \psi_{,\gamma}\lambda v^\gamma$$

$$\psi(x^\gamma+\lambda v^\gamma) = \phi(x^\gamma+\varepsilon\xi^\gamma+\lambda \cancel{v}^\gamma)$$

$$\psi(x^\gamma) \qquad \psi(x^\gamma) = C_1$$

$$\frac{\partial\psi}{\partial x^\gamma}\lambda v^\gamma = \frac{\partial\phi}{\partial x^\gamma}\,\varepsilon\xi^\gamma + \lambda v^\gamma$$

$$\phi(x^\gamma+\varepsilon\xi^\gamma(x^\gamma)) = \psi(x^\gamma)$$

$$\phi(x^\gamma) + \phi_{,\gamma}\,\varepsilon\xi^\gamma = \psi(x^\gamma)$$

$$\psi(x^\gamma) = \phi\left(x^\gamma + \varepsilon\xi^\gamma(x^\gamma)\right) = \phi(y^\alpha)$$

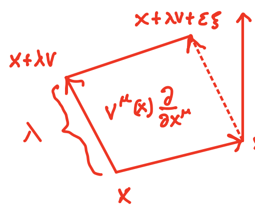
$$\xi^\gamma\left[D_\gamma \psi(x^\gamma)\right] = \frac{\partial\phi}{\partial y^\alpha}\frac{\partial\left(x^\alpha + \varepsilon\xi^\alpha(x)\right)}{\partial x^\gamma} = \left[\left.\frac{\partial\phi}{\partial y^\alpha}\right|_{y=x+\varepsilon\xi}\left(\delta^\alpha_\gamma + \varepsilon\,\xi^\alpha_{,\gamma}\right)\right]\xi^\gamma(x)$$

$$= \left.\frac{\partial\phi}{\partial y^\alpha}\right|_{y=x+\varepsilon\xi}\xi^\alpha$$

V^γ

$$v^\mu(x^\gamma + \varepsilon\xi^\gamma)\left.\frac{\partial\psi(y)}{\partial y^\gamma}\right|_{y=x+\varepsilon\xi} = v^\mu(x^\gamma)\frac{\partial}{\partial x^\alpha}\psi(x+\varepsilon\xi)$$

$$V^\mu(x^\gamma + \varepsilon\xi^\gamma)\frac{\partial}{\partial x^\mu}$$



$$V^\mu(x^\gamma + \varepsilon\xi^\gamma)\frac{\partial}{\partial x^\mu}$$

V^μ

$$\psi(x+\lambda V) = \psi(x) + \left.\frac{\partial\psi}{\partial x^\gamma}\right|_x \lambda V^\gamma(x)$$

$$\psi(x+\varepsilon\xi) = \psi(x) + \left.\frac{\partial\psi}{\partial x^\alpha}\right|_x \varepsilon\xi^\alpha(x)$$

$$\psi(x+\lambda V+\varepsilon\xi) = \psi(x+\lambda V) + \left.\frac{\partial\psi}{\partial x^\gamma}\right|_{x+\lambda V}\varepsilon\xi^\gamma(x+\lambda V)$$

$$= \psi(x) + \frac{\partial\psi}{\partial x^\gamma}\lambda V^\gamma + \left.\frac{\partial\psi}{\partial x^\gamma}\right|_x \varepsilon\xi^\gamma(x)$$

$$+\frac{1}{2}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial\psi}{\partial x^\gamma}\lambda V^\gamma\right)\lambda V^\gamma + \frac{\partial}{\partial x^\alpha}\left[\left.\frac{\partial\psi}{\partial x^\gamma}\right|_x \varepsilon\xi^\gamma(x)\right]\lambda V^\alpha$$

$$\psi(x+\varepsilon\xi+\lambda V) = \psi(x+\varepsilon\xi) + \left[\frac{\partial\psi}{\partial x^\alpha}\xi^\alpha\right]_{x+\varepsilon\xi}\lambda V^\alpha(x+\varepsilon\xi)$$

$$= \psi(x) + \frac{\partial\psi}{\partial x^\gamma}\varepsilon\xi^\gamma + \left[\frac{\partial\psi}{\partial x^\alpha}\xi^\alpha\right]_x \lambda V^\alpha(x)$$

$$+\frac{1}{2}\frac{\partial}{\partial x^\alpha}\left[\frac{\partial\psi}{\partial x^\gamma}\varepsilon\xi^\gamma\right]\varepsilon\xi^\alpha + \frac{\partial}{\partial x^\gamma}\left[\frac{\partial\psi}{\partial x^\alpha}\xi^\alpha\right]_x \lambda V$$

$$\left\langle dx'^\gamma, \frac{\partial}{\partial x'^\beta}\right\rangle = \left\langle \frac{\partial x'^\gamma}{\partial x^\alpha}dx^\alpha, \frac{\partial x^\sigma}{\partial x'^\beta}\frac{\partial}{\partial x^\sigma}\right\rangle = \left\langle\left(\delta^\gamma_\alpha + \varepsilon\,\xi^\gamma_{,\alpha}(x)\right)dx^\alpha, \left(\delta^\sigma_\beta - \varepsilon\,\xi^\sigma_{,\beta}(x')\right)\frac{\partial}{\partial x^\sigma}\right\rangle$$

$$= \left[\delta^\gamma_\alpha\,\delta^\sigma_\beta + \varepsilon\,\xi^\gamma_{,\alpha}(x)\,\delta^\sigma_\beta - \delta^\gamma_\alpha\,\varepsilon\,\xi^\sigma_{,\beta}(x') - \varepsilon\varepsilon\,\xi^\gamma_{,\alpha}(x)\,\xi^\sigma_{,\beta}(x')\right]\left\langle dx^\alpha, \frac{\partial}{\partial x^\sigma}\right\rangle$$

$$\xi^\mu_\theta \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \theta} \text{ and } \xi^\mu_\varphi \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \varphi}$$

generated transformations are symmetry transformations. This is because they leave each of the following geometrical and physical tensor fields in spacetime invariant.

1. Metric tensor:

$$g_{\mu\nu}\, dx^\mu dx^\nu = g_{AB}(x^C)\, dx^A dx^B + \overbrace{r^2(x^C)\, \underbrace{(d\theta^2 + \sin^2\theta\, d\varphi^2)}_{\gamma_{ab}\, dx^a dx^b}}^{g_{ab}\, dx^a dx^b} : \left[\begin{array}{c|cc} g_{AB} & \begin{matrix}0 & 0\\ 0 & 0\end{matrix} \\ \hline \begin{matrix}0 & 0\\ 0 & 0\end{matrix} & r^2\gamma_{ab} \end{array}\right]$$

$g_{AB}(x^C)\, dx^A dx^B$

$r^2(x^C)$

$\gamma_{ab}\, dx^a dx^b = d\theta^2 + \sin^2\theta\, d\varphi^2$

2. Momenergy tensor:

$$t_{\mu\nu}\, dx^\mu dx^\nu = t_{AB}(x^C)\, dx^A dx^B + \overbrace{\underbrace{t(x^C)\, r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)}_{t\, g_{ab}\, dx^a dx^b}}^{t_{ab}\, dx^a dx^b} : \left[\begin{array}{c|cc} t_{AB} & \begin{matrix}0 & 0\\ 0 & 0\end{matrix} \\ \hline \begin{matrix}0 & 0\\ 0 & 0\end{matrix} & t\, g_{ab} \end{array}\right]$$

$t_{AB}(x^C)\, dx^A dx^B$

$t(x^C)$ (= "transverse pressure into the θ and φ directions")

3. General covector:

$$v_\mu\, dx^\mu : \begin{bmatrix} v_A \\ 0 \\ 0 \end{bmatrix}$$

4. The Klein-Gordon field equation

$$\Box\psi - \frac{m^2c^2}{\hbar^2}\psi \equiv \left(g^{\mu\nu}\frac{\partial\psi}{\partial x^\mu}\right)_{;\nu} - \lambda_c^{-2}\psi$$

$$= \left(g^{AB}\frac{\partial\psi}{\partial x^A}\right)_{|B} + \frac{1}{r^2}\left(\gamma^{ab}\frac{\partial\psi}{\partial x^a}\right)_{:b} - \frac{1}{\lambda^2}\psi$$

$$= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^A}\left[\sqrt{-g}\, g^{AB}\frac{\partial\psi}{\partial x^B}\right] + \frac{1}{r^2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial^2\psi}{\partial\varphi^2}\right\} - \frac{1}{\lambda_c^2}\psi$$

Let $\psi = \psi_{\ell m}(x^a)\, Y_\ell^m(\theta,\varphi)$ be a spherical normal mode solution. It satisfies

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial^2\psi}{\partial\varphi^2} = -\ell(\ell+1)\psi$$

The amplitude $\psi_{\ell m}(x^a)$ satifies

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^A}\left[\sqrt{-g}\, g^{AB}\frac{\partial\psi_{\ell m}}{\partial x^B}\right] - \left(\frac{\ell(\ell+1)}{r^2} + \frac{1}{\lambda_c^2}\right)\psi_{\ell m} = 0$$

4. Einstein tensor

$$G_{\mu\nu}\, dx^\mu dx^\nu \equiv \left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R\right) dx^\mu dx^\nu$$

$$= G_{AB}\, dx^A dx^B + \underbrace{G_{ab}}_{\frac{1}{2} G_d{}^d g_{ab}} dx^a dx^b \qquad \left[\begin{array}{c|c} G_{AB} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & G_{ab} \end{array}\right]$$

$$G_{AB} = \frac{1}{r^2}\left\{ -2\, r_{;A|B} + \left(2 r r_{;c}{}^{|c} + r_{;c} r^{;c} - 1\right) g_{AB} \right\}$$

$$G_{ab} \equiv \frac{1}{2} G_d{}^d g_{ab} = \left(\frac{r_{;c} r^{;c}}{r} - \mathcal{R}\right) g_{ab}$$

where $\mathcal{R}$ is the Gaussian curvature defined by

$$^{(2)}R^{AB}{}_{CD} = \mathcal{R}\left(\delta^A_C \delta^B_D - \delta^A_D \delta^B_C\right)$$

5. The Einstein field equations

$$G_{AB} = -2 r\, r_{;A|B} + g_{AB}\left(2 r r_{;c}{}^{|c} + r_{;C} r_{;D}\, g^{CD} - 1\right) = \frac{8\pi G}{c^2} r^2 t_{AB}$$

$$\frac{1}{2} G_a{}^a = \frac{r_{;C|D}\, g^{CD}}{r} - \mathcal{R} = \frac{8\pi G}{c^2} t$$

6. The conservation equation

$G_\mu{}^\nu{}_{;\nu} \equiv 0 = t_\mu{}^\nu{}_{;\nu}$ leads to

$$\left(r^2 G_A{}^B\right)_{|B} - r\, r_{;A}\, G_a{}^a = 0$$

which implies

$$\left(r^2 t_A{}^B\right)_{|B} - r\, r_{;A}\, 2t = 0,$$

the conservation of momenergy, Euler hydrodynamics for a spherically symmetric system.