Lecture 32 & 33

Integration of the Einstein Field Equations using a conservation law

I. The 2+2 Decomposition of Spacetime with Spherical Symmetry

Given a spherically symmetric system, split its dynamics, (which is governed by the E.F.E.s) into two subsystems:

One coordinatized by the spherical coordinates on the 2-sphere S^2, the "transverse manifold";
the other by a radial and time coordinate, the "longitudinal manifold" M^2. The result of this split is that the spacetime manifold M^4 got factored into two submanifolds:

$$M^4 = M^2 \times S^2.$$

The metric $ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu$ in the representation which reflects this decomposition is

$$ds^2 = g_{AB}(x^a)\,dx^A dx^B + r^2(x^a)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

Thus, all spherically symmetric gravitational systems are mathematized by two degrees of freedom on M^2,

$$g_{AB}(x^a)\,dx^A dx^B \qquad (\text{"metric tensor field on } M^2\text{"})$$

$$r^2(x^a) \qquad (\text{scalar field on } M^2)$$

32.2

The Einstein field equations $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} \underbrace{t_{\mu\nu}}$ have the same type of decomposition:

t_{00} in units of energy density

$$r^2 G_{AB} = -2r\, r_{;A|B} + g_{AB}\left(2r\, r_{;c}{}^{|c} + r_{;c}\, r_{;D}\, g^{CD} - 1\right) = \frac{8\pi G}{c^2} r^2 t_{AB} \qquad (32.1)$$

("tensor field equation on M^2)

$$G_{ab} = \left(\frac{r_{;C|D}\, g^{CD}}{r} - \mathcal{R}\right) r^2 \gamma_{ab} = \frac{8\pi G}{c^2} t\, r^2 \gamma_{ab} \qquad (32.2)$$

("scalar field equation on M^2)

II. Partial integration of the E.F.E.s for spherically symmetric system.

The gravitational field equations can be integrated in part by combining the Bianchi identity*

$$\left(r^2 G_A{}^B\right)_{|B} - r\, r_{;A}\left(\frac{r_{;C|D}\, g^{CD}}{r} - \mathcal{R}\right) = 0$$

with the E.F.E., Eq.(32.1)

* \footnote{ which is the only vectorial identity on M^2, which resulted from the 2+2 decomposition of the Bianchi identidy $G_\mu{}^\nu{}_{;\nu} = 0$.}

The process consists of 3 steps.

32.3

Step 1.

Multiply $r^2 G_A{}^B$ by $-\frac{1}{2} r_{;C}\, \epsilon^{CA}$, obtain the M^2 vector

$$-\frac{1}{2} r_{;C}\, \epsilon^{CA}\, r^2\, G_A{}^B \equiv J^B, \tag{32.3}$$

and find that its divergence vanishes:

$$J^B{}_{|B} = 0. \tag{32.4}$$

Step 2.

The divergence condition, Eq. (32.4), implies that there exist a scalar ψ on M^2 with a gradient whose components

$$\psi_{,E} = -J^B\, \epsilon_{BE}\,. \quad (\text{"conservative" vector field on } M^2)$$

This conclusion is based on integrating the ordinary first order differential equation implied by Eq. (32.4).

Indeed, recall that the covariant divergence of any vector field can always be expressed in terms of an ordinary divergence

$$0 = J^B{}_{|B} \equiv \frac{1}{\sqrt{-g}}\, \frac{\partial\left(\sqrt{-g}\, J^B\right)}{\partial x^B}$$

$$= \frac{1}{\sqrt{-g}} \left[\underbrace{\frac{\partial\left(\sqrt{-g}\, J^0\right)}{\partial x^0}}_{\equiv\ \partial M/\partial x^0} + \underbrace{\frac{\partial\left(\sqrt{-g}\, J^1\right)}{\partial x^1}}_{\equiv\ \partial(-N)/\partial x^1} \right].$$

Here

and
$$M = \sqrt{-g}\,J^0$$
$$N = -\sqrt{-g}\,J^1 \qquad (32.5)$$

The vanishing of $J^B{}_{|B} = 0$ guarantees that $M\,dx^1 + N\,dx^0$ is an exact differential.

"Exact differential" means that there exists a scalar ψ such that

$$M\,dx^1 + N\,dx^0 = \frac{\partial \psi}{\partial x^1}dx^1 + \frac{\partial \psi}{\partial x^0}dx^0 \qquad (32.6)$$

This is because $J^B{}_{|B} = 0$ implies that

$$\frac{\partial M}{\partial x^0} = \frac{\partial N}{\partial x^1}.$$

It follows that

$$\frac{\partial \psi}{\partial x^1} = M = \sqrt{-g}\,J^0$$

and

$$\frac{\partial \psi}{\partial x^0} = N = -\sqrt{-g}\,J^0$$

or

$$\boxed{\frac{\partial \psi}{\partial x^E} = -J^B \epsilon_{BE}} \qquad (32.7)$$

Step 3.

Find the scalar ψ by applying the boxed Eq. (32.7) to Eq. (32.3).

$$\frac{\partial \psi}{\partial x^E} = \frac{1}{2} r_{,C}\, \epsilon^{CA}\, r^2\, G_A{}^B \epsilon_{BE} \left(= -J^B \epsilon_{BE}\right) \qquad (32.8)$$

Insert Eq. (32.1), the expression for

$$r^2 G_A{}^B = -2r\, r_{;A}{}^{|B} + \delta_A{}^B \left(2r\, r_{;C}{}^{|C} + r_{;C}\, r_{;D}\, g^{CD} - 1\right),$$

simplify and find

$$\frac{\partial \Psi}{\partial X^E} = \left[\frac{1}{2} r\left(1 - r_{;D}\, r^{|D}\right)\right]_{,E} \qquad (32.9)$$

The 3-step mathematical deducton that the $G_A{}^B$ expression on the right hand side of Eq. (32.8) is a conservative vector field is a step forward in integrating the E.F.E.s. The scalar whose gradient is this field is by inspection

$$\Psi = \frac{1}{2} r\left(1 - r_{;C}\, r_{;D}\, g^{CD}\right). \qquad (32.10)$$

The r-r coefficient of the inverse metric is therefore

$$g^{rr} = 1 - \frac{2\Psi}{r}. \qquad (32.11)$$

III. Conservation of Spherical Mass-energy.

The physical meaning of the scalar function $\Psi(x^0, x^1 = r)$ is furnish by the E.F.E.s.

On one hand the r.h.s. of Eq. (32.9) features the gradient of this scalar, on the other hand Eq. (32.8) is, via the E.F.E.s, proportional to the momergy density,

$$\frac{\partial \Psi}{\partial x^E} = -\frac{1}{2} r^2 r_{;C} \epsilon^{CA} \frac{8\pi G}{c^4} t_A{}^B \epsilon_{BE}. \qquad (32.12) \quad 32.6$$

The equality of these two mathematizes the conservation of gravitational mass-energy. No matter how violent and complex the spherical process, that mass-energy is conserved.

Being the gradient of a scalar, the line integral of the r.h.s. of Eq. (32.12),

$$\Psi(x^0; x^1) = \int^{(x^0; x^1)} \frac{\partial \Psi}{\partial x^E} dx^E = -\frac{1}{2} \int^{(x^0; x^1)} r_{;C} \epsilon^{CA} r^2 \frac{8\pi G}{c^4} t_A{}^B \epsilon_{BE}\, dx^E, \qquad (32.13)$$

is independent of its path between two fixed events in M^2, the 2-d longitudinal spacetime manifold.* The

\footnote{Physically each of its events is associated with a sphere of area $4\pi r^2(x^0; x^1)$.}

integral depends only on its end point-events. The integral vanish over a closed path.

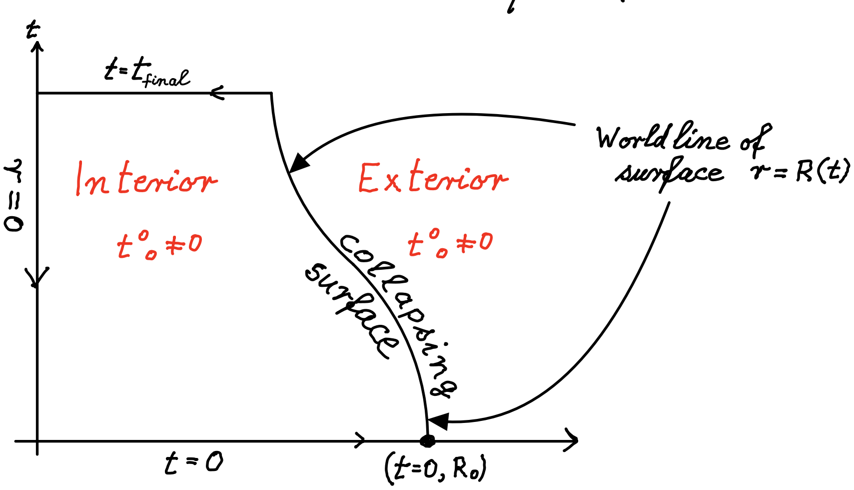

Figure 32.1 A closed contour integral whose initial ($t=0$) and final ($t=t_{final}$) spatial line integral yield the conserved mass M

The integrand vanishes at $x^1=r=0$.

It also vanishes beyond the matter-vacuum interface:

For $r>R(t)$, where $t_0{}^0=0$.

$$t_0{}^0(t,r)=\begin{cases}\neq 0 & r<R(t) \quad \text{INSIDE}\\ =0 & R(t)<r \quad \text{OUTSIDE}\end{cases}$$

$$\psi(t=0,r=R_0)=\psi\left(t=t_{fin},R(t_{fin})\right)$$

$$=-\frac{4\pi G}{c^4}\int_0^{R(t)}\epsilon^{r0}\,r^2\,t_0{}^0\,\epsilon_{0r}\,dr$$

The $t_0{}^0$ component of the momenergy tensor is $(-)\rho c^2$, the negative of the mass-energy density,

$$t_0{}^0=(-)\rho c^2,$$

where ρ is the mass density. Thus

$$\psi(t,R(t)) = \frac{G}{c^2}\int_0^{R(t)}4\pi r^2\rho(r)\,dr=\frac{G}{c^2}\cdot$$

$$=\frac{G}{c^2}\times\begin{pmatrix}\text{Conserved mass}\\ \text{enclosed by a}\\ \text{collapsing sphere}\end{pmatrix}$$

$$=\frac{G}{c^2}\,m(t,R(t))$$

Comment

1. $g^{rr}=1-\frac{2G}{c^2}m(t,r)$, inverse metric coefficient, Eq. (32.11)

(32.8)

2. $\psi(r;t) = \frac{G}{c^2}\, m(r;t)$ has units of length, Eq. (32.10)

It is mass expressed in geometrical units.

For example: $M_\odot = 2\times 10^{33}\,\text{gr} = 1.5\,\text{km}$

$M_\oplus = 6\times 10^{27}\,\text{gr} = .44\,\text{cm}$