# Lecture 33 & 34

Non-Euclidean Geometry in the
Equatorial Plane of a Star

In MTW's chapter 23 read Section 23.8

# I. Geometry of spacetime for a static star:



The spacetime geometry for any spherically symmetric system has the form

$$ds^2 = -e^{2\phi(r,t)}dt^2 + \frac{dr^2}{1-\frac{m(r,t)}{r}} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

For a system which is also static, there is no time dependence. Its spatial geometry at any fixed time is therefore

$$ds^2\Big|_{t=\text{fixed}} = \frac{dr^2}{1-\frac{2m(r)}{r}} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

In the equatorial plane $\theta = \frac{\pi}{2}$ it is

$$ds^2\Big|_{\substack{t=\text{fixed}\\ \theta=\frac{\pi}{2}}} \equiv d\sigma^2 = \frac{dr^2}{1-\frac{2m(r)}{r}} + r^2\, d\varphi^2 \qquad \text{(Non-Euclidean)}$$

which is to be compared with

$$ds^2 = dr^2 + r^2 d\varphi^2 \qquad \text{(Euclidean)}$$

## A. Imbedding Space

To obtain a geometrical picture of this non-Euclidean geometry, use the method of the imbedding diagram according to which one views the non-Euclidean plane as a surface of revolution in a 3-d fictitious imbedding space with a Euclidean geometry and spanned by its three coordinates $z$, $r$, and $\varphi$:

$$dl^2 = dz^2 + dr^2 + r^2 d\varphi^2 \qquad \text{("metric for the imbedding space")}$$

On the to-be-found surface of revolution

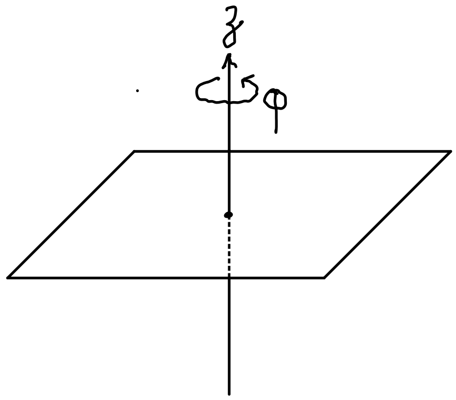


Figure 33.1 Fictitious 3-d imbedding space induces a non-Euclidean surface of revolution.

$z = f(r)$

the ambient Euclidean geometry induces the metric

$$dl^2\Big|_{z=f(r)} = \left[\left(\frac{dz}{dr}\right)^2 + 1\right] dr^2 + r^2 d\varphi^2 . \qquad \text{("metric for the surface of revolution")}$$

## B. The Imbedding Function

Identify the metric on the to-be-found surface of revolution with the metric on the equatorial plane of the spherically symmetric spacetime. This results in the differential equation

$$\left(\frac{dz}{dr}\right)^2 + 1 = \frac{1}{1 - \frac{2m(r)}{r}}$$

The solution to this differential equation

$$z(r) = \int_0^r \left[\frac{2m(r')}{r' - 2m(r')}\right]^{1/2} dr' + \text{const.} \qquad (33.1)$$

yields a 2-d surface of revolution from a 3-d perspective.

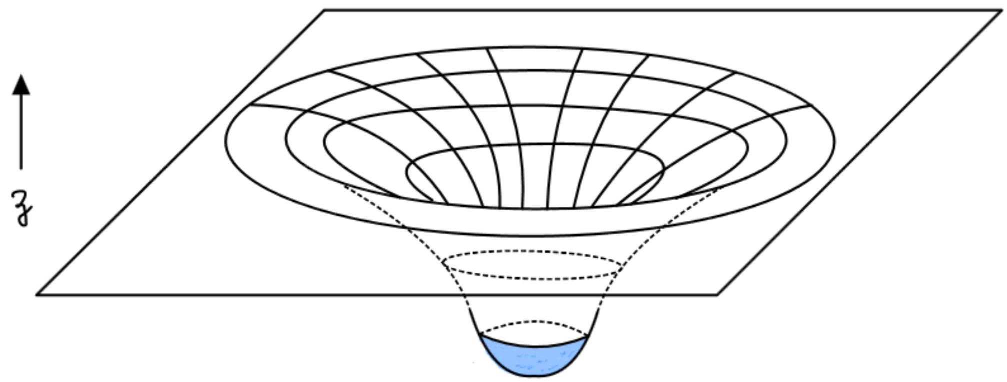


Figure 33.2 Imbedding diagram for the equatorial plane of a homogeneous star.

It allows one to visualize the inner 2-d spatial geometry on the equatorial plane or – because of spherical symmetry – any other rotated plane of the spherically symmetric space.

## C. Example

Consider at some fixed time ($t=$ const.) a star with mass density $\rho(r)$ in its interior and vacuum on the outside.

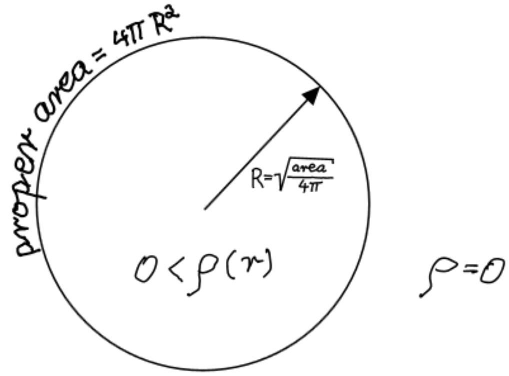

Figure 33.3 The radial parameter R for the concentric spheres of the star's interior and exterior is quantified in term of their proper area: $R = \sqrt{\text{area}/4\pi}$.

For such a configuration the mass function and its associated imbedding function are

$$m(r) = \begin{cases} \int_0^r 4\pi r'^2 \rho(r')\,dr' & \text{inside: } r < R \\ M & \text{outside: } r > R \end{cases}$$

and

$$z(r) = \begin{cases} \int_0^r \left[\frac{2m(r')}{r-2m(r')}\right]^{1/2} dr' + c & \text{inside: } r<R \quad (33.2) \\ \left[8M(r-2M)\right]^{1/2} + c & \text{outside: } r>R \quad (33.3) \end{cases}$$

Comment

Here the mass $M$ and the mass density $\rho(r)$ are expressed in term of geometrical units:

$$M = \frac{G}{c^2} M^{\text{conventional}} = \left[\frac{G}{c^2}(\text{mass})\right] = [\text{length}]$$

$$\rho = \frac{G}{c^2}\rho^{\text{conventional}} = \left[\frac{G}{c^2}\frac{(\text{mass})}{(\text{length})^3}\right] = \left[\frac{1}{(\text{length})^2}\right]$$

a) Thus outside the star one has

$$(z-c)^2 = 8M(r-2M)$$

which is a parabola of revolution.

b) Inside the star, near the center

$$m(r) = \frac{4\pi\rho_c}{3} r^3$$

The geometrized mass density has units $\frac{1}{(\text{length})^2}$.
Consequently, the density $\rho_c$ implies a geometrically determined standard of length designated by $\underline{a}$:

$$\frac{8\pi}{3}\rho_c \equiv \frac{1}{a^2}.$$

With this scale factor the imbedding function brings into sharp focus the essence the nature of the geometry in the central region inside the star:

$$z = \int_0^r \sqrt{\frac{\left(\frac{r'}{a}\right)^2}{1-\left(\frac{r'}{a}\right)^2}}\, dr' = -a\sqrt{1-\left(\frac{r'}{a}\right)^2}\,\Bigg|_0^r$$

$$= a - \sqrt{a^2 - r^2} \qquad \text{for } r \ll a, \text{ near the center}$$

Thus the imbedding function $z(r)$ is part of the circle of revolution:

$$(z-a)^2 + r^2 = a^2$$

c) At the star's boundary

$$\frac{dz}{dr} = \sqrt{\frac{2m(r)}{r-2m(r)}}$$

is continuous because $m(r)$ is continuous.

The geometry of a star is therefore characterized by a circle of revolution near its center, and a parabola of revolution outside its interior joined to its surface $r = R$

without any kinks. This is because $m(r)$ is continuous there. Figure 33.4 depicts via, the imbedding diagram for a homogeneous star, the equatorial plane with its non-Euclidean geometry.

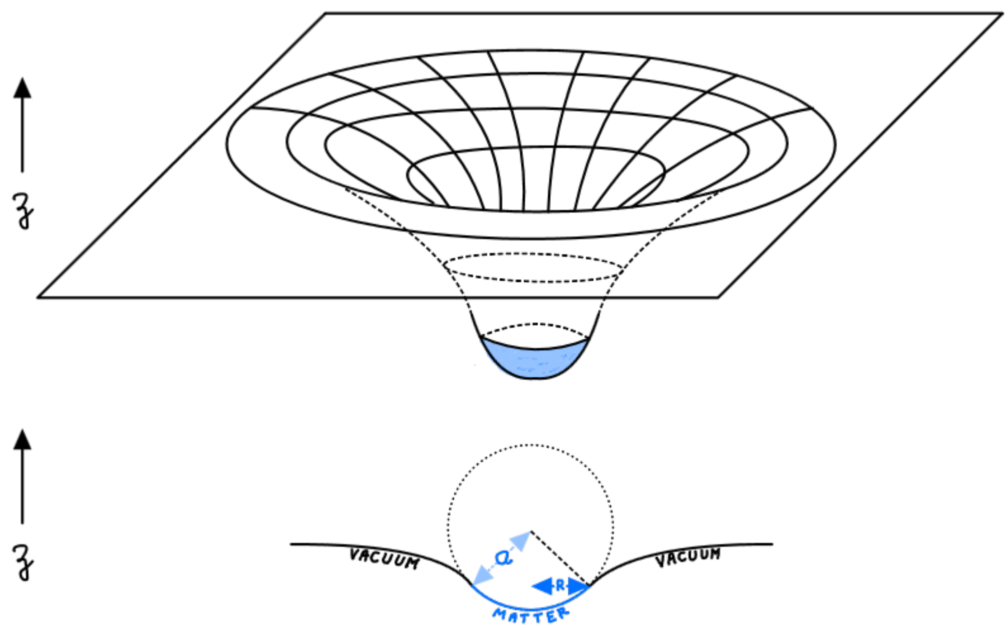


Figure 33.4 The imbedding for the equatorial plane, here the one of a homogeneous star, highlights its non-Euclidean nature both inside and outside the star.

The non-Euclidean nature of the equatorial plane inside the star has its basis in the physical world by comparing two lengths measurements.

1. From the surface of the star drill a hole all the way to its center. Drop a plumb line from the surface to the center. Its proper length $\ell$ is

$$\ell = \int_0^R \sqrt{g_{rr}}\, dr = \int_0^R \frac{dr}{\sqrt{1-\frac{2m(r)}{r}}} > R$$

2. Measure the proper equatorial circumference of the star. Given that its surface area is $4\pi R^2$, that circumference $\mathcal{C}$ is

$$\mathcal{C} = \int_0^{2\pi} R\, d\varphi = 2\pi R$$

It is a fact that there is a mismatch between the determination of the circumference based on radial measurements and that based on circum-navigating the star, i.e. that

$$\frac{2\pi \ell}{\mathcal{C}} = \frac{\int_0^R \frac{dr}{\sqrt{1-\frac{2m(r')}{r'}}}}{R} > 1 ,$$

This is due to the non-Euclidean nature of the equatorial plane of the star.