Lecture 35

Relativistic Stars

In MTW's chapter 23 read Sections 23.2-23.7

I. SPHERICALLY SYMMETRIC SYSTEMS

It is a fact that there exists a multitude of gravitating system which are spherically symmetric.

How does one classify them?

All such systems are governed by the Einstein field equations adapted to spherical symmetry

$$-2r\, r_{,A}{}^{|B} + \delta_A{}^B\left(2r\, r_{,c}{}^{|c} + r_{,c}\, r^{|c} - 1\right) \equiv r^2 G_A{}^B = \frac{8\pi G}{c^4} r^2\, t_A{}^B \qquad (35.1)$$

$$\left(\frac{r_{,c}{}^{|c}}{r} - R\right)\delta_a{}^b \equiv G_a{}^b = \frac{8\pi G}{c^4}\, t\, \delta_a{}^b \qquad (35.2)$$

together with the implied hydrodynamicl Euler equations of motion

$$\left(r^2 t_A{}^B\right)_{|B} - r\, r_{,A}\, t = 0. \qquad (35.3)$$

Here

$$\left[t_\mu{}^\nu\right] = \left[\begin{array}{c|c} t_A{}^B & 0 \\ \hline 0 & t\,\delta_a{}^b \end{array}\right],$$

are the components of the momenergy tensor relative to the coordinate frame which reflects the rotational symmetries of the metric tensor field.

How does one distinguish such gravitating systems?

Achieve this task by applying it solving the above equations for a particular spherical star.

II. RELATIVISTIC STAR

Consider a spherical self-gravitating system consisting of a perfect fluid (no viscosity!). The components of its momenergy are (see Lecture 16)

$$t_\mu{}^\nu = (p+\rho)\, u_\mu u^\nu + p\, \delta_\mu{}^\nu = \begin{cases} T_A{}^B = (p+\rho)\, u_A u^B + p\, \delta_A^B \\ T_a{}^b = p_\perp \delta_a^b \text{ : xverse pressure} \end{cases}$$

Here p, ρ, and u^μ are the pressure, energy density, and 4-velocity components of the fluid. Their distribution in the star is governed by the law of momenergy conservation

$$t_\mu{}^\nu{}_{;\nu} = 0 \,; \quad u^\mu t_\mu{}^\nu{}_{;\nu} = 0$$

These are the equations that govern the dynamics of a relativistic fluid.

For a spherically symmetric configuration there is a single vectorial equation on $M^2 = M^4/S^2$:

$$t_C{}^\nu{}_{;\nu} - u_C u^B t_B{}^\nu{}_{;\nu} =$$

$$= u_{C|B} u^B (p+\rho) - \left(\delta_C{}^B + u_C u^B\right) \frac{\partial p}{\partial x^B} = 0 \qquad C = 0, 1 \qquad (35.4)$$

The metric for *any* spherically symmetric configuration (by an appropriate choice of coordinates) has the diagonal form

$$g_{\mu\nu}\,dx^\mu dx^\nu = \underbrace{-e^{2\Phi(t,r)}dt^2 + e^{2\Lambda(t,r)}dr^2}_{g_{AB}\,dx^A dx^B} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Focus on a star in equilibrium. Consequently, there is no explicit time dependence in all matter and geometrical variables. In particular

$$p = p(r)\,;\ \rho = \rho(r)\,;\ \{u^A\} = \{u^0, u^1 = 0\}$$

Accordingly, the two components of the vectorial hydrodynamical Eqs. (35.4) yield only one:

For $c=0$ one has $0=0$

For $c=1$ one has $\dfrac{dp}{dr} = -\dfrac{d\Phi}{dr}(p+\rho)$ (35.5)

Furthermore, the tensorial Einstein field Eq.(35.1) with its three components yields

$$r^2 G_0{}^0 \equiv -2\frac{\partial m}{\partial r} \qquad = \frac{8\pi G}{c^4} r^2\, t_0{}^0 \; \left(= -\frac{8\pi G}{c^2} r^2 \rho\right) \tag{35.6a}$$

$$r^2 G_{01} \equiv 2r\frac{\partial \Lambda}{\partial t} \qquad = \frac{8\pi G}{c^4} r^2\, t_{01} \; (= \quad 0 \quad) \tag{35.6b}$$

$$r^2 G_1{}^1 \equiv 2(r-2m)\frac{\partial \Phi}{\partial r} - \frac{m}{r} = \frac{8\pi G}{c^4} r^2\, t_1{}^1 \; \left(= \frac{8\pi G}{c^4} r^2 p\right) \tag{35.6c}$$

For a system in equilibrium these equations imply

$$r^2 G_0{}^0: \quad \boxed{\frac{dm}{dr} = \frac{4\pi G}{c^2}\rho} \tag{35.7}$$

$$r^2 G_{01}: \quad \dot{\Lambda} = 0 \tag{35.8}$$

$$r^2 G_1{}^1: \quad \boxed{\frac{d\Phi}{dr} = \frac{m + (4\pi G/c^4) r^3 p}{r(r-2m)}} \tag{35.9}$$

Insert the expression for $\frac{d\Phi}{dr}$, Eq. (35.7c) into Eq. (35.5). The result is

$$\boxed{\frac{dp}{dr} = -\frac{m + (4\pi G/c^2) r^3 p}{r(r-2m)}(p+\rho)} = -\frac{G}{c^2}\frac{m^{conv} + (4\pi r^3 p)/c^2}{r^2\left(1-\frac{2m}{r}\right)}(p+\rho) \tag{35.10}$$

The three boxed equations form a coupled system of non-linear ordinary differential equations:

a) <u>two</u> for the <u>gravitational degrees of freedom,</u> $m(r)$ and $\Phi(r)$,

b) <u>one</u> for the <u>matter degree of freedom.</u> $p(r)$.

These equations govern any static spherically symmetric perfect fluid configuration.

However, in order to keeps one's mathematical connected to the world around us, one must follow the dictum that a diffential equation is never solved until one imposes boundary conditions on its solution. For the determination of the structure of the star the equations (35.7), (35.9), and (35.10) need to be augmented by specifying

(i) the star's central density,

$$\rho(r=0)=\rho_c ,$$

(ii) an equation of state, $p=p(\rho)$, throughout the star so so that

$$p(r=0)=p(\rho_c),$$

and

(iii) the fact that the star has a center; i.e. that

$$m(r=0)=0 ;$$

Otherwise the pressure gradient, Eq. (35.10), will not be finite at $r=0$.

III. HOW TO SOLVE THE EQUATIONS OF HYDROSTATIC EQUILIBRIUM

The structure of a star in equilibrium is determined by

$$\boxed{\frac{dp}{dr} = -\frac{(m+4\pi r^3 p)}{r(r-2m)}(p+\rho) \quad \text{with } p(r=0)=p_c}$$

Within a Newtonian framework this equation expresses a balance between a force due to a pressure gradient and the gravitational force acting on a small volume of fluid in the star, namely $\frac{dp}{dr} = -\frac{m}{r^2}\rho$

The mass enclosed in a sphere of surface area $4\pi r^2$ is

$$\boxed{m(r) = \int_0^r 4\pi \rho\, r^2 dr \qquad \text{with } m(0)=0}$$

These two equations together with an equation of state

$$\rho = \rho(p)$$

determine the equilibrium structure of the star.

To find the structure of a star integrate from the center $r=0$ (where we must $m(0)=0$ so that the pressure gradient $\frac{dp}{dr}$ and hence p stays finite at $r=0$) outward until we get to that radius, call it $r=R$, where the pressure vanished:

$\boxed{\text{b.c. for } R}$ is $\boxed{p(R)=0}$ $\boxed{\text{surface of the star}}$

At $r=R$

$\boxed{m(R)=M}$ "Total mass" of the star.

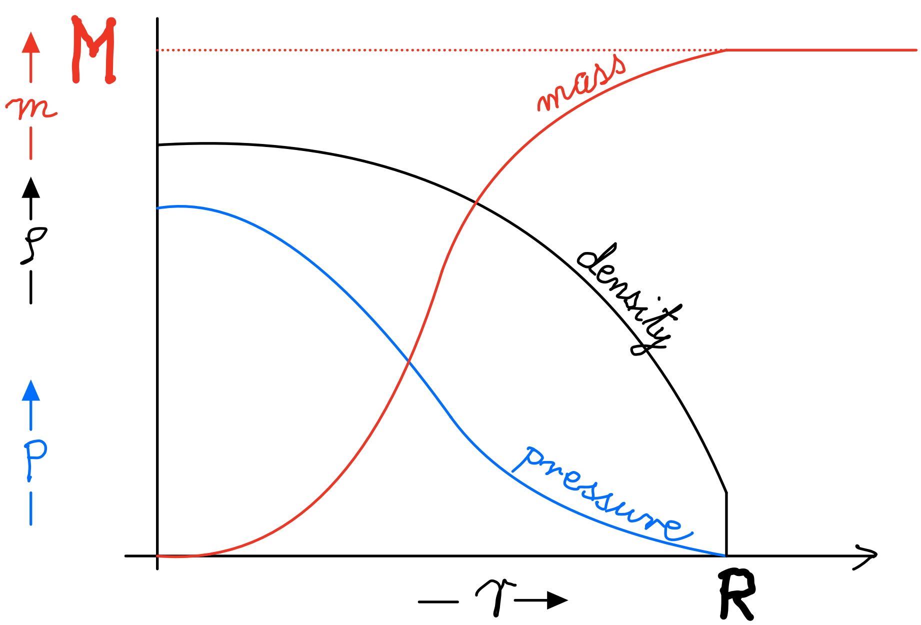

Figure 35.1 Qalitative depiction of a solution to the structure equations, namely, the density $\rho(r)$, pressure $p(r)$, and massfunction $m(r)$ of a star. If the star has

a crust, its surface density would be discontinuous, even though its density is zero at the surface.

IV. EXTERNAL GRAVITATIONAL FIELD OF A STAR

A. Outside the star, where $p=0, \rho=0$ we have

1. $m(r) = m(R) = M$ constant outside

thus

$$\boxed{g_{rr} = \frac{1}{1-\frac{2M}{r}}} \qquad r>M \qquad \text{outside}$$

2. $p=0$ outside (USE Eq. (35.9) on P 35.4.

$\therefore \quad \frac{d\Phi}{dr} = \frac{M}{r^2\left(1-\frac{2M}{r}\right)} = \frac{1}{2}\frac{d}{dr}\ln\left(1-\frac{2M}{r}\right)$ $\quad \ln\left\{\left(1-\frac{2M}{r}\right)\cdot C\right\}$

subject to $\Phi(r=\infty)=0$

(i) $-g_{tt} = e^{2\Phi} = C\left(1-\frac{2M}{r}\right)$

where we have to chose that integration constant $C=1$, which assures us that the boundary condition

$$\boxed{\Phi(r=\infty)=0}$$

is satisfied.

ii. Newtonian correspondence limit compels us to call $M\left(=M_{conv}\frac{G}{c^2}\right)$ the mass of the star — the mass which determines the planetary orbits.

B. Metric outside any spherical star:

$$\boxed{ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)}$$