Lecture 36

H-J Theory of Particle Mechanics: Asymptotic solutions to Wave equation

In MTW Read BOX 25.3 (The philosophy illustrated)

Peruse Exercise 35.15 (The math)

Read § 25.5 (The application)

36.1

I. Wave Equation in the Geometrical Optics Approximation

The geometry of gravition controls the geodesic motion of free particles. The fact that Plank's constant ($\hbar = 6.6 \times 10^{-34}$ mks $= 6.6 \times 10^{-27}$ cgs units) is not zero implies that the laws of particle motion are to be mathematized in terms of the Klein-Gordon (K-G) equation

$$-\frac{\partial^2 \psi}{\partial t^2} + \nabla^2 \psi - \frac{m^2}{\hbar^2}\psi = g^{\mu\nu}\,\psi_{,\mu;\nu} - \frac{m^2}{\hbar^2} = 0 \qquad (36.1)$$

for relativistic mechanics.* For non-relati-

* \footnote{ For an electron the Compton wave length is $\frac{\hbar}{m_e c} = .4 \times 10^{-10}$ cm }

vistic mechanics let

$$\psi = e^{i m t/\hbar}\,\varphi$$

and find for $\frac{m}{\hbar}|\varphi| \gg \left|\frac{\partial \varphi}{\partial t}\right|$ that the non-relativistic Schrödinger wave equation in a weak gravitational potential ($\frac{\Phi_{grav'l}}{c^2} \ll 1$; see Eq. (5.9)) is

$$\left[\frac{-\hbar^2}{2m}\nabla^2 + m\phi_{grav'l}\right]\varphi = i\hbar\frac{\partial \varphi}{\partial t}. \qquad (36.2)$$

These wave equations govern the manner in which the wave function evolves in spacetime, which in general is not flat. (36.2)

Both the Schrödinger equation and Newton's equation of motion

$$m\,\frac{d\vec{x}}{dt^2}=\vec{F},$$

which governs the dynamics of a non-relativistic particle, have its mass as an arbitrary parameter. The same observation is true for the K-G equation and the geodesic equation of a relativistic particle.

Hamilton-Jacobi (H-J) theory takes advantage of this observation by furnishing the logical connecting link between (i) wave mechanics and particle mechanics, (ii) physical optics and geometrical optics as well as between (iii) Fourier theory and theory of wave packets.

36.3

Both relativistic and non-relativistic wave functions can be reexpressed in terms of a phase function S ("eikonal", "Schrödinger phase", "dynamical phase") and an amplitude $\mathcal{A}$:

$$\psi(x^\alpha) = \mathcal{A}(x^\alpha)\, e^{iS(x^\alpha)/\hbar}.$$

Here, in the asymtodical short wave length / high frequency limit

$\mathcal{A}(x^\alpha)$ = "slowly varying function of x^α"

$e^{iS(x^\alpha)/\hbar}$ = "rapidly " " of x^α"

Introduce such a function into the wave equation (36.1) and find

$$g^{\mu\nu}\left\{ \mathcal{A}_{,\mu;\nu} + \frac{2i}{\hbar}\frac{\partial S}{\partial x^\mu}\frac{\partial \mathcal{A}}{\partial x^\nu} + \frac{i}{\hbar} S_{,\mu;\nu}\mathcal{A} - \frac{1}{\hbar^2}\left[\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}\mathcal{A}\right] - \frac{m^2}{\hbar^2}\mathcal{A}\right\} e^{iS/\hbar} = 0 \qquad (36.2)$$

The wave equation (36.1), and, hence Eq. (36.2), applies to all masses m. Mathematize this observation by introducing the dimensionless variable ϵ into Eq. (36.1) and any of its solutions by letting

$$m = \frac{m_0}{\epsilon}.$$

Thus

$$\epsilon^2 g^{\mu\nu}\psi_{,\mu;\nu} - \frac{m_0^2}{\hbar^2}\psi = 0 \qquad (36.3)$$

and

$$\psi = \left(\mathcal{A}_0 + \mathcal{A}_1\epsilon + \mathcal{A}_2\epsilon^2 + \cdots\right) e^{i\frac{1}{\epsilon}S/\hbar} \qquad (36.4)$$

Using H-J theory we shall show that in the asymptotic limit, $\epsilon \to 0$, which corresponds to very heavy particles, (36.4)

$$\frac{m_0}{\epsilon} \longrightarrow \infty,$$

$\psi(x^\alpha)$ furnishes us with the solution $\{x^\mu(\tau)\}_{\mu=0}^{3}$ to the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \qquad \mu = 0,1,2,3$$

without having to go through the labor of having to solve this system of o.d.e.'s.

In other words, the asymptotic limit of wave dynamics consists of the mechanics of particles executing their space-time trajectories.

Apply Eq.(36.4) to (36.2), collect equal powers of ϵ, and set their coefficients to zero:

36.5

$\frac{1}{\epsilon^2}$: $$\frac{\mathcal{A}_0}{\hbar^2}\left\{ g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + m_0^2\right\} = 0$$

$\frac{1}{\epsilon}$: $$\frac{i}{\hbar}\left\{ \left(g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\right)_{;\mu}\mathcal{A}_0 + 2g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\mathcal{A}_{0;\nu}\right\} = 0$$

(Terms of higher order yields "post geometrical optics" corrections.)

Dropping the subscript "zero" from m_0, one therefore obtains:

$$g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + m^2 = 0 \qquad \text{(H-J eq'n)} \qquad (36.5)$$

$$\left(\mathcal{A}_0^2\, g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\right)_{;\nu} = 0 \qquad \left(\begin{matrix}\text{Particle}\\ \text{conservation}\end{matrix}\right) \qquad (36.6)$$

The application of this asymptotic expansion method to the (non-relativistic) Schrödinger equation yields

$$\frac{1}{2m}\vec{\nabla}S\cdot\vec{\nabla}S + U(x) + \frac{\partial S}{\partial t} = 0 \qquad (36.7)$$

$$\frac{\partial(\mathcal{A}_0^2)}{\partial t} + \frac{1}{m}\vec{\nabla}\cdot\left(\mathcal{A}_0^2\vec{\nabla}S\right) = 0 \qquad (36.8)$$

Equations (36.5) and (36.7) are the H-J equations for a relativistic and non-relativistic system respectively.

They are first (36.6)
order partial differential equations
whose solutions are the dynamical
phase S of the given system.
Once it is known for a given system,
the task of exhibiting its global
spacetime particle trajectories in
mathematical form is complete.
This is because of the application
of the principle of constructive
interference to the phase function
is mathematically trivial (although
physically fundamental).

Eqs (36.6) & (36.8), both of which are (36.7)
4-dimensional divergence conditions,
mathematize the law of conservation
of particles. Since each particle
carries a certain amount of
momenergy, one finds that this
law plays a key role in mathe-
matizing the law of momenergy
conservation in terms of the
energy-momentum tensor of an
aggregate of particles.

Summary:

In the "geometrical optics" limit of wave mechanics the wave length of a wave is so short that locally the phase fronts have the (straight and parallel) properties of a plane wave.

This is so despite the fact that these phase fronts exhibit curvature outside any local neighborhood.

The shape and spacing of the phase fronts is expressed by the isograms of phase function $S(x^\alpha)$.

It satifies the H-J equation

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial x}, x, t\right) = 0 \quad \text{(non-relativistic)} \tag{9}$$

or

$$\mathcal{H}\left(\frac{\partial S}{\partial x^\alpha}, x^\alpha\right) \equiv g^{\mu\nu}(x^\alpha)\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\beta} + m^2 = 0 \quad \text{(relativistic)} \tag{10}$$

If the particle has charge q and is moving in an electro-magnetic field has the electomagnetic vector potential $A_\mu(x^\alpha)$, then its H-J equation is

$$\mathcal{H}\left(\frac{\partial S}{\partial x^\alpha}, x^\alpha\right) \equiv g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu} - qA_\mu\right)\left(\frac{\partial S}{\partial x^\nu} - qA_\nu\right) + m^2 = 0$$