Lecture 38

Reconstruction of classical worldlines from the Principle of Constructive Interference

I. Relativistic H-J equation and its solutions

The reconstruction of classical worldlines of particles via the application of the principle of constructive interference can be generalized to any system characterized by an action, and hence by an Hamiltonian.

Consider the H-J equation

$$\mathcal{H}\left(x^\alpha; \frac{\partial S}{\partial x^\mu}\right) \equiv g^{\mu\nu}(x^\alpha)\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + m^2 = 0 \qquad (38.1)$$

for a free particle in an environment coordinatized by global rectilinear coordinates. In such an environment the inverse metric is independent of each coordinate, $t, x, y,$ and z. They are termed "cyclic" coordinates, and the H-J equation is simply

$$-\left(\frac{\partial S}{\partial t}\right)^2 + \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 + m^2 = 0 \qquad (38.2)$$

Apply the method of the separation of variables* first to x,

$$S = X(x) + S'(t, y, z)$$

$$\left(\frac{dX}{dx}\right)^2 = \left(\frac{\partial S'}{\partial t}\right)^2 - \left(\frac{\partial S'}{\partial y}\right)^2 - \left(\frac{\partial S'}{\partial z}\right)^2 - m^2 = p_x^2 \qquad (\equiv \text{"separation constant"})$$

and find that

$$S = p_x x + S'(t, y, z) + \text{const.}$$

The resulting principle is this:

Whenever the H-J equation has cyclic coordinate, its solution is a linear function of this coordinate.

Applying this principle to the y and z coordinates results in

$$S = p_x x + p_y y + p_z z + T(t) + \text{const.}$$

Thus

$$p_x^2 + p_y^2 + p_z^2 + m^2 = \left(\frac{dT}{dt}\right)^2$$

which implies that** the dynamical phase is

$$S = p_x x + p_y y + p_z z - \sqrt{p_x^2 + p_y^2 + p_z^2 + m^2}\, t + \beta(p_x, p_y, p_z) \qquad (38.3)$$

* \footnote{ Had one started by first separating t,

$$S = T(t) + S''(x, y, z),$$

one would have found that the dynamical phase is

$$S = -p_0 t + p_x x + p_y y \pm \sqrt{p_0^2 - p_x^2 - p_y^2 - m^2}\, z + \gamma(p_0, p_x, p_y).\}$$

** \footnote{ The minus sign in front of the square-root has been chosen in order that the phase velocity 4-vector

$$\left\{\eta^{\nu\mu}\frac{\partial S}{\partial x^\mu}\right\}_{\nu=0}^{3} = \left\{-\sqrt{p_x^2 + p_y^2 + p_z^2 + m^2}, p_x, p_y, p_z\right\}$$

points into the future.}

Consider the H-J equation in the static environment of a spherical system,

$$-\frac{1}{1-\frac{2M}{r}}\left(\frac{\partial S}{\partial t}\right)^2+\left(1-\frac{2M}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2+\frac{1}{r^2}\,\frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2+m^2=0. \qquad (38.4)$$

For this dynamical system t and φ are cyclic coordinates, while θ and r are not. This H-J equation is soluble by the method of the separation of variable. Solutions such as these,

$$S=S\left(x^0, x^1, x^2, x^3; \alpha_1, \alpha_2, \alpha_3\right)+\beta(\alpha_1, \alpha_2, \alpha_3) \equiv S(x^\mu; \alpha_i)+\beta(\alpha_i), \qquad (38.5)$$

if one can find them, always have three separation/integration constants — constants that refer to the essential properties of the dynamical system:

$$(\alpha_1, \alpha_2, \alpha_3)=\begin{cases}(p_x, p_y, p_z)\\ \left(\sqrt{p_x^2+p_y^2+p_z^2+m^2},\, p_y, p_z\right)\\ \left(\text{energy},\ \begin{matrix}\text{total}\\ \text{angular}\\ \text{momentum}\end{matrix},\ \begin{matrix}\text{azimuthal}\\ \text{angular}\\ \text{momentum}\end{matrix}\right)\end{cases}$$

II. Constructive Interference.

Mathematically, constructive interference is based on the condition that

$$\psi(x^\mu)=\iiint_{\{(\alpha_1,\alpha_2,\alpha_3)\}} \underbrace{\mathcal{A}(x^\mu; \alpha_i)}_{\substack{\text{slowly}\\ \text{varying}}}\ \underbrace{e^{iS(x^\mu;\alpha_i)/\hbar}}_{\substack{\text{radidly}\\ \text{varying}}}\ d\alpha_1\, d\alpha_2\, d\alpha_3 \qquad (38.6)$$

represent a localized wavepacket whose maximum intensity $|\psi|^2_{max}$ traces out a worldline in spacetime. This maximum occurs (38.4) wherever the phase of $e^{iS/\hbar}$ is stationary in the α_1-α_2-α_3-space.

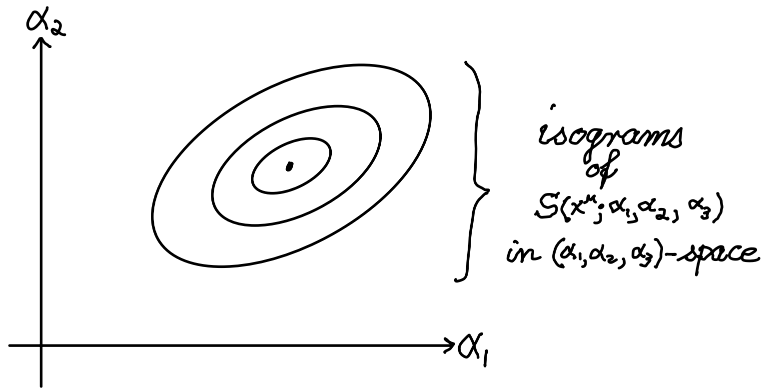

Figure 38.1 The integral $\psi(x^\mu)$ gets it dominant contribution from the neighborhood surrounding the critical point of S in $(\alpha_1, \alpha_2, \alpha_3)$-space.

The conditions which guarantee this are

$$\left.\begin{aligned} 0 &= \frac{\partial S}{\partial \alpha_1} = \frac{\partial \mathcal{S}(x^0, x^1, x^2; \alpha_i)}{\partial \alpha_1} + \frac{\partial B}{\partial \alpha_1} \\ 0 &= \frac{\partial S}{\partial \alpha_2} = \frac{\partial \mathcal{S}(x^0, x^1, x^2; \alpha_i)}{\partial \alpha_2} + \frac{\partial B}{\partial \alpha_2} \\ 0 &= \frac{\partial S}{\partial \alpha_3} = \frac{\partial \mathcal{S}(x^0, x^1, x^2; \alpha_i)}{\partial \alpha_3} + \frac{\partial B}{\partial \alpha_3} \end{aligned}\right\} \quad (38.7)$$

Each of these conditions is the equation for a 3-d manifold in the 4-d spacetime. Their intersection is a 1-d trajectory, a geodesic in spacetime.

III. Geodesic Equations

The tangent $\mathbf{u}$ to this 1-d trajectory lies in the intersection of these three manifolds of stationary phase.

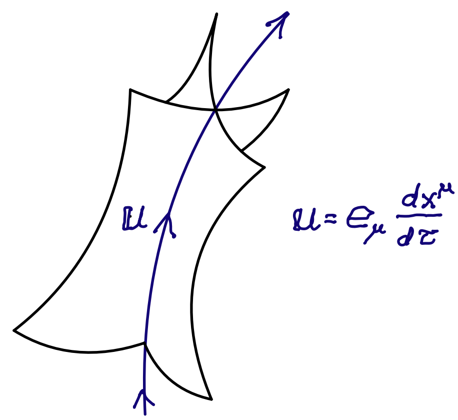

Figure 38.2 Geodesic as the intersection of the surface of stationary phase

Consequently, $\frac{\partial S}{\partial \alpha_i}$ is constant along this 1-d trajectory

$$\frac{dx^\mu}{d\tau}\frac{\partial}{\partial x^\mu}\left(\frac{\partial S}{\partial \alpha_1}\right)=0 \qquad \frac{dx^\mu}{d\tau}\frac{\partial}{\partial x^\mu}\left(\frac{\partial S}{\partial \alpha_2}\right)=0 \qquad \frac{dx^\mu}{d\tau}\frac{\partial}{\partial x^\mu}\left(\frac{\partial S}{\partial \alpha_3}\right)=0$$

Equivalently on has

$$\begin{bmatrix} \frac{\partial^2 S}{\partial x^0 \partial \alpha_1} & \frac{\partial^2 S}{\partial x^1 \partial \alpha_1} & \frac{\partial^2 S}{\partial x^2 \partial \alpha_1} & \frac{\partial^2 S}{\partial x^3 \partial \alpha_1} \\ \frac{\partial^2 S}{\partial x^0 \partial \alpha_2} & \frac{\partial^2 S}{\partial x^1 \partial \alpha_2} & \frac{\partial^2 S}{\partial x^2 \partial \alpha_2} & \frac{\partial^2 S}{\partial x^3 \partial \alpha_2} \\ \frac{\partial^2 S}{\partial x^0 \partial \alpha_3} & \frac{\partial^2 S}{\partial x^1 \partial \alpha_3} & \frac{\partial^2 S}{\partial x^2 \partial \alpha_3} & \frac{\partial^2 S}{\partial x^3 \partial \alpha_3} \end{bmatrix} \begin{bmatrix} \frac{dx^0}{d\tau} \\ \frac{dx^1}{d\tau} \\ \frac{dx^2}{d\tau} \\ \frac{dx^3}{d\tau} \end{bmatrix} = [0] \qquad (38.8)$$

On the other hand the H-J Eq.(38.1) is

$$\mathcal{H}\Big(x^\nu ; \underbrace{\frac{\partial S}{\partial x^\mu}(x^\nu; \alpha_1,\alpha_2,\alpha_3)}_{\displaystyle \equiv p_\mu}\Big)=0 \quad \text{for all } \alpha_1, \alpha_2, \alpha_3 .$$

Thus

$$\frac{\partial}{\partial \alpha_i}\mathcal{H} = \frac{\partial}{\partial \alpha_i}\left(\frac{\partial S}{\partial x^\mu}\right)\frac{\partial \mathcal{H}}{\partial p_\mu} = 0 \qquad i=1,2,3$$

or equivalently, since mixed partial are equal,

$$\begin{bmatrix} \frac{\partial^2 S}{\partial x^0 \partial\alpha_1} & \frac{\partial^2 S}{\partial x^1 \partial\alpha_1} & \frac{\partial^2 S}{\partial x^2 \partial\alpha_1} & \frac{\partial^2 S}{\partial x^3 \partial\alpha_1} \\ \frac{\partial^2 S}{\partial x^0 \partial\alpha_2} & \frac{\partial^2 S}{\partial x^1 \partial\alpha_2} & \frac{\partial^2 S}{\partial x^2 \partial\alpha_2} & \frac{\partial^2 S}{\partial x^3 \partial\alpha_2} \\ \frac{\partial^2 S}{\partial x^0 \partial\alpha_3} & \frac{\partial^2 S}{\partial x^1 \partial\alpha_3} & \frac{\partial^2 S}{\partial x^2 \partial\alpha_3} & \frac{\partial^2 S}{\partial x^3 \partial\alpha_3} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial p_0} \\ \frac{\partial \mathcal{H}}{\partial p_1} \\ \frac{\partial \mathcal{H}}{\partial p_2} \\ \frac{\partial \mathcal{H}}{\partial p_3} \end{bmatrix} = [0] \qquad (38.9)$$

As before,
we assume that the set $\{\alpha_1, \alpha_2, \alpha_3\}$ is a complete set of integration constants, i.e. that there is no functional relation between them. This fact is mathematized by the statement that the 3×4 matrix $\left[\frac{\partial^2 S}{\partial x^\mu \partial \alpha_i}\right]$ has maximal rank, i.e. its null space is 1-dimensional

$$\dim \mathcal{N}\left(\left[\frac{\partial^2 S}{\partial x^\mu \partial \alpha_i}\right]\right) = 1 .$$

It follows that the nullspace solution to Eq.(38.9) has a unique direction also, namely

$$\frac{\partial \mathcal{H}}{\partial p_\mu} = M \epsilon^{\mu\alpha\beta\gamma} \frac{\partial}{\partial x^\alpha}\left(\frac{\partial S}{\partial \alpha_1}\right) \frac{\partial}{\partial x^\beta}\left(\frac{\partial S}{\partial \alpha_2}\right) \frac{\partial}{\partial x^\gamma}\left(\frac{\partial S}{\partial \alpha_3}\right) \qquad (39.10)$$

where M is a proportionality factor.

Combining Eqs. (38.7) and (38.8) results in

$$\boxed{\frac{dx^\mu}{d\tau} = N(\tau)\frac{\partial \mathcal{H}}{\partial p_\mu}} \quad \left(= N(\tau)\, \epsilon^{\mu\alpha\beta\gamma} \left(S_{,\alpha_1}\right)_{,\alpha} \left(S_{,\alpha_2}\right)_{,\beta} \left(S_{,\alpha_3}\right)_{,\gamma}\right)$$

which is the 1st half of the Hamilton's equations of motion.

The arbitrariness in the τ-dependent proportionality factor expresses the indeterminateness in the parametrization of the curve.

Having established the direction of the tangent at one point, we now ask and answer about changes in the momentum $p_\mu = \frac{\partial S}{\partial x^\mu}$ as one proceeds along the world line.

The fact that the H–J holds everywhere implies

$$0 = \frac{\partial}{\partial x^\nu} \mathcal{H}\Big(\underbrace{\frac{\partial S(x^\sigma, \alpha_i)}{\partial x^\mu}}_{\equiv p_\mu}, x^\nu\Big)$$

$$= \frac{\partial}{\partial x^\nu}\left(\frac{\partial S}{\partial x^\mu}\right) \left.\frac{\partial \mathcal{H}}{\partial p_\mu}\right)_{x^\nu} + \left.\frac{\partial \mathcal{H}}{\partial x^\nu}\right)_{p_\mu}$$

With the help of

$$\frac{\partial \mathcal{H}}{\partial p_\mu} = \frac{1}{N} \frac{dx^\mu}{d\tau}$$

we obtain

$$0 = \frac{\partial}{\partial x^\mu}\underbrace{\left(\frac{\partial S}{\partial x^\nu}\right)}_{p_\nu} \frac{1}{N} \frac{dx^\mu}{d\tau} + \left.\frac{\partial \mathcal{H}}{\partial x^\nu}\right)_{p_\mu} = 0$$

or

$$\boxed{\frac{dp_\nu}{d\tau} = -N(\tau) \frac{\partial \mathcal{H}}{\partial x^\nu}}$$

This equation together with

$$\boxed{\frac{dx^\mu}{d\tau} = N(\tau) \frac{\partial \mathcal{H}}{\partial p_\mu}}$$

are Hamilton's equations of motion.

Applied to

$$\mathcal{H} = g^{\alpha\beta} p_\alpha p_\beta + m^2$$

they imply that

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} - \frac{1}{N}\frac{dN}{d\tau}\frac{dx^\mu}{d\tau} = 0$$

which is the equation for a geodesic if we choose a parametrization $d\lambda = N\, d\tau$.

Consistency demands that $\mathcal{H} = 0$ be satisfied along the whole worldline. This can be verified from the fact that

$$\frac{d}{d\tau}\mathcal{H} = 0$$

i.e. $\mathcal{H} = \text{const}$, is a consequence of the Hamilton's equations of motion.