Lecture 39

Particle orbits in a static, spherically symmetric spacetime environment

Read in MTW Box 25.4, and Section 25.5

I. The ubiquity and depth of H-J theory in physics

In physics momenergy manifests itself in the form of the dynamics of particles and fields. Considering their diverse manifestation, it is difficult to find a perspective that provides a wider conceptual unification than the mathematical physics perspective of H-J theory. The commonality in its manifestations can be summarized by the symbolic equation

$$\text{H-J theory} = \begin{pmatrix}\text{particle}\\ \text{mechanics}\end{pmatrix} \cap \begin{pmatrix}\text{wave}\\ \text{mechanis}\end{pmatrix} \cap \begin{pmatrix}\text{geometrical}\\ \text{optics}\end{pmatrix} \cap \begin{pmatrix}\text{wave}\\ \text{optics}\end{pmatrix} \cap \begin{pmatrix}\text{classical}\\ \text{relativistic}\\ \text{mechanics}\end{pmatrix} \cap \begin{pmatrix}\text{relatistic}\\ \text{quantum}\\ \text{mechanics}\end{pmatrix}$$

II. H-J theory for the mechanics of a particle in the Schwartzschild geometry.

A. H-J equation and its solution

The metric for the Schwarzschild geometry is

$$g_{\mu\nu}\,dx^\mu dx^\nu = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{39.1}$$

The corresponding H-J equation is (39.2)

$$0 = g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + m^2 = -\frac{1}{1-\frac{2M}{r}}\left(\frac{\partial S}{\partial t}\right)^2 + \left(1-\frac{2M}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2 + m^2 \tag{39.2}$$

Its prominent feature is that t and φ are cyclic coordinates. Consequently, separation of variables yields

$$\frac{\partial S}{\partial t} = \text{const} \equiv -E$$

$$\frac{\partial S}{\partial \varphi} = \text{const.} \equiv p_\varphi ,$$

and therefore

$$S = -Et + p_\varphi \varphi + S'(r,\theta) .$$

It follows that the H-J equation reduces to a p.d.e. of two variables only

$$-\frac{E^2}{1-\frac{2M}{r}} + \left(1-\frac{2M}{r}\right)\left(\frac{\partial S'}{\partial r}\right)^2 + \frac{1}{r^2}\left[\left(\frac{\partial S'}{\partial \theta}\right)^2 + \frac{p_\varphi^2}{\sin^2\theta}\right] + m^2 = 0 .$$

Continuing with the separation of variables process, set

$$S'(r,\theta) = R(r) + \Theta(\theta)$$

and isolate the expression which depends on θ only:

$$\left[\left(\frac{d\Theta}{d\theta}\right)^2 + \frac{p_\varphi^2}{\sin^2\theta}\right] = \frac{r^2}{\left(1-\frac{2M}{r}\right)}\left\{E^2 - m^2\left(1-\frac{2M}{r}\right) - \left(1-\frac{2M}{r}\right)^2\left(\frac{dR}{dr}\right)^2\right\} .$$

Both sides of this reworked H-J equation are equal to the same (obviously) non-negative constant $\ell^2 \geq 0$.

It follows that

$$\frac{d\Theta}{d\theta} = \pm\sqrt{\ell^2 - \frac{p_\varphi^2}{\sin^2\theta}} \tag{39.3}$$

$$\frac{dR}{dr} = \frac{\pm 1}{\left(1-\frac{2M}{r}\right)}\left\{E^2 - \left(\frac{\ell^2}{r^2}+m^2\right)\left(1-\frac{2M}{r}\right)\right\}^{1/2}$$

The complete solution to the H-J equation is therefore

$$S(t,r,\theta,\varphi) = \underbrace{\int^t -E\,dt'}_{p_t} \pm \underbrace{\int^r \left\{E^2 - \left(\frac{\ell^2}{r'^2}+m^2\right)\left(1-\frac{2M}{r'}\right)\right\}^{1/2}\frac{1}{\left(1-\frac{2M}{r'}\right)}}_{p_r}\,dr' \pm \int^\theta \underbrace{\sqrt{\ell^2 - \frac{p_\varphi^2}{\sin^2\theta'}}}_{p_\theta}\,d\theta' + \int^\varphi \underbrace{p_\varphi}_{p_\varphi}\,d\varphi' + \beta(E,\ell^2,p_\varphi^2) \tag{39.4}$$

This globally defined dynamical phase is in the form of a path-independent line integral. Its gradient is the 4-momentum covector

$$d S = \frac{\partial S}{\partial x^\mu}\,\underset{\sim}{d}x^\mu = p_\mu(x^\alpha)\,\underset{\sim}{d}x^\mu.$$

B. The conditions for constructive interference.

A dynamical phase function is characterized by three separation/integration constants:

$$\alpha_1 = -E \qquad \text{(− "mass-energy")}$$
$$\alpha_2 = \ell^2 \qquad \text{("[angular momentum]}^2\text{")}$$
$$\alpha_3 = p_\varphi \qquad \text{("z-component of the angular momentum")}$$

Constructive interference applied to a dynamical phase function, Eq.(39.4), yields

$$0=\frac{\partial S}{\partial E}=-t+\int\frac{E}{\pm\left[E^2-\left(\frac{\ell^2}{r^2}+m^2\right)\left(1-\frac{2M}{r}\right)\right]^{1/2}}\frac{dr}{\left(1-\frac{2M}{r}\right)}+\frac{\partial\beta}{\partial E} \tag{39.5}$$

$$0=\frac{\partial S}{\partial \ell^2}=\int\frac{\frac{1}{2}\,d\theta}{\pm\sqrt{\ell^2-\frac{p_\varphi^2}{\sin^2\theta}}}-\int\frac{1/2}{\pm\left[E^2-\left(\frac{\ell^2}{r^2}+m^2\right)\left(1-\frac{2M}{r}\right)\right]^{1/2}}\frac{dr}{r^2}+\frac{\partial\beta}{\partial \ell^2} \tag{39.6}$$

$$0=\frac{\partial S}{\partial p_\varphi}=\int\frac{p_\varphi}{\pm\sqrt{\ell^2-\frac{p_\varphi^2}{\sin^2\theta}}}\frac{d\theta}{\sin^2\theta}+\varphi+\frac{\partial\beta}{\partial p_\varphi} \tag{39.7}$$

For a given set of integration (= separation) constants

$$E,\ \ell^2,\ p_\varphi;\ \frac{\partial\beta}{\partial E},\ \frac{\partial\beta}{\partial\ell^2},\ \frac{\partial\beta}{\partial p_\varphi} \tag{39.8}$$

each of these three interference conditions defines a 3-dimensional submanifold in the ambient 4-d spacetime spanned by its (t, r, θ, φ) coordinate system.

The intersection of these submanifolds is a specific 1-d submanifold, the globally defined particle worldline. Each one is uniquely identified by the six parameters, Eq. (39.8)

The tangents to these worldlines are determined by

$$\frac{dx^\mu}{d\tau}=\frac{\partial\mathcal{H}}{\partial p_\mu}\qquad \mu=0,1,2,3 \tag{39.9}$$

where $\mathcal{H}=\frac{1}{2}g^{\mu\nu}p_\mu p_\nu$

$$\frac{dt}{d\tau} = \frac{E}{1-\frac{2M}{r}} = -\frac{1}{1-\frac{2M}{r}}\frac{\partial S}{\partial t} \quad (39.10)$$

$$\frac{dr}{d\tau} = \left[E^2 - \left(1-\frac{2M}{r}\right)\left(\frac{\ell^2}{r^2}+m^2\right)\right]^{1/2} = \left(1-\frac{2M}{r}\right)\frac{\partial S}{\partial r} \quad (39.11)$$

$$\frac{d\theta}{d\tau} = \frac{1}{r^2}\left[\ell^2 - \frac{p_\varphi^2}{\sin^2\theta}\right]^{1/2} = \frac{1}{r^2}\frac{\partial S}{\partial \theta} \quad (39.12)$$

$$\frac{d\varphi}{d\tau} = \frac{1}{r^2}\frac{p_\varphi}{\sin^2\theta} = \frac{1}{r^2\sin^2\theta}\frac{\partial S}{\partial \varphi} \quad (39.13)$$

The constructive interference conditions Eqs. (39.5)-(39.7) on page 39.4 do not lack any geometrical and physical information about the dynamics of free particle in the Schwarzschild geometry represented relative to the metric as represented by

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2\left[d\theta^2 + \sin^2\theta\, d\varphi^2\right].$$

However, instead of giving a mathematically completed analysis based on Eqs (39.5)-(39.7) one can already draw important conclusions based on the requirement that classically (i.e., not wavemechanically) the particle satisfy

$$\left(\frac{\partial S}{\partial r}\right)^2 \geq 0\ , \left(\frac{\partial S}{\partial \theta}\right)^2 \geq 0. \qquad (39.14)$$

C. Classically allowed vs classically forbidden regions

Because of inequalities Eqs. (39.14), space is divided into regions which a classically allowed vs. those which are classically forbidden. There $\left(\frac{\partial S}{\partial r}\right)^2 < 0$ and $\left(\frac{\partial S}{\partial \theta}\right)^2 < 0$, meaning that the momentum components become imaginary! The boundary between the regions is located where

$$\boxed{\left(\frac{\partial S}{\partial r}\right)^2 = 0} \quad \boxed{\left(\frac{\partial S}{\partial \theta}\right)^2 = 0}$$

The significance of this boundary one infers from the Hamilton's equations of motion Eqs. (39.9). They imply that

$$\frac{dr}{d\tau} = \frac{\partial \mathcal{H}}{\partial p_r} = g^{rr}\frac{\partial S}{\partial r} = \pm\left[E^2 - \left(\frac{\ell^2}{r^2} + m^2\right)\left(1 - \frac{2M}{r}\right)\right]^{1/2}$$

$$\frac{d\theta}{d\tau} = \frac{\partial \mathcal{H}}{\partial p_\theta} = g^{\theta\theta}\frac{\partial S}{\partial \theta} = \pm\frac{1}{r^2}\sqrt{\ell^2 - \frac{p_\varphi^2}{\sin^2\theta}}$$

Thus the boundary between what is classically allowed and what is forbidden is the locus of points where the radial and polar angle motion, comes to a momentary halt:

$$\frac{dr}{d\tau}=0$$

and

$$\frac{d\theta}{d\tau}=0.$$

This is the location of <u>turning points</u>, where particle motions $\frac{dr}{d\tau}$ and $\frac{d\theta}{d\tau}$ must reverse sign.

D. The effective potential for classically allowed motion.

From this locus of turning points one can infer major qualitative aspects such as bounded vs unbounded motion, stable vs unstable motion.

As an example, consider the radial motion as determined by its locus of turning points:

$$\frac{dr}{d\tau}=0 \Rightarrow E^2-V^2_{eff}(r)=0$$

Upon considering equatorial motion $\theta=\frac{\pi}{2}$ one has $\ell^2=p_\varphi^2$ so that

$$V^2_{eff}=m^2-\frac{2M}{r}m^2+\frac{p_\varphi^2}{r^2}-\frac{2M}{r}\frac{p_\varphi^2}{r^2}$$
$$=m^2\left(1-\frac{2M}{r}\right)\left(1+\frac{p_\varphi^2}{r^2}/m^2\right)$$

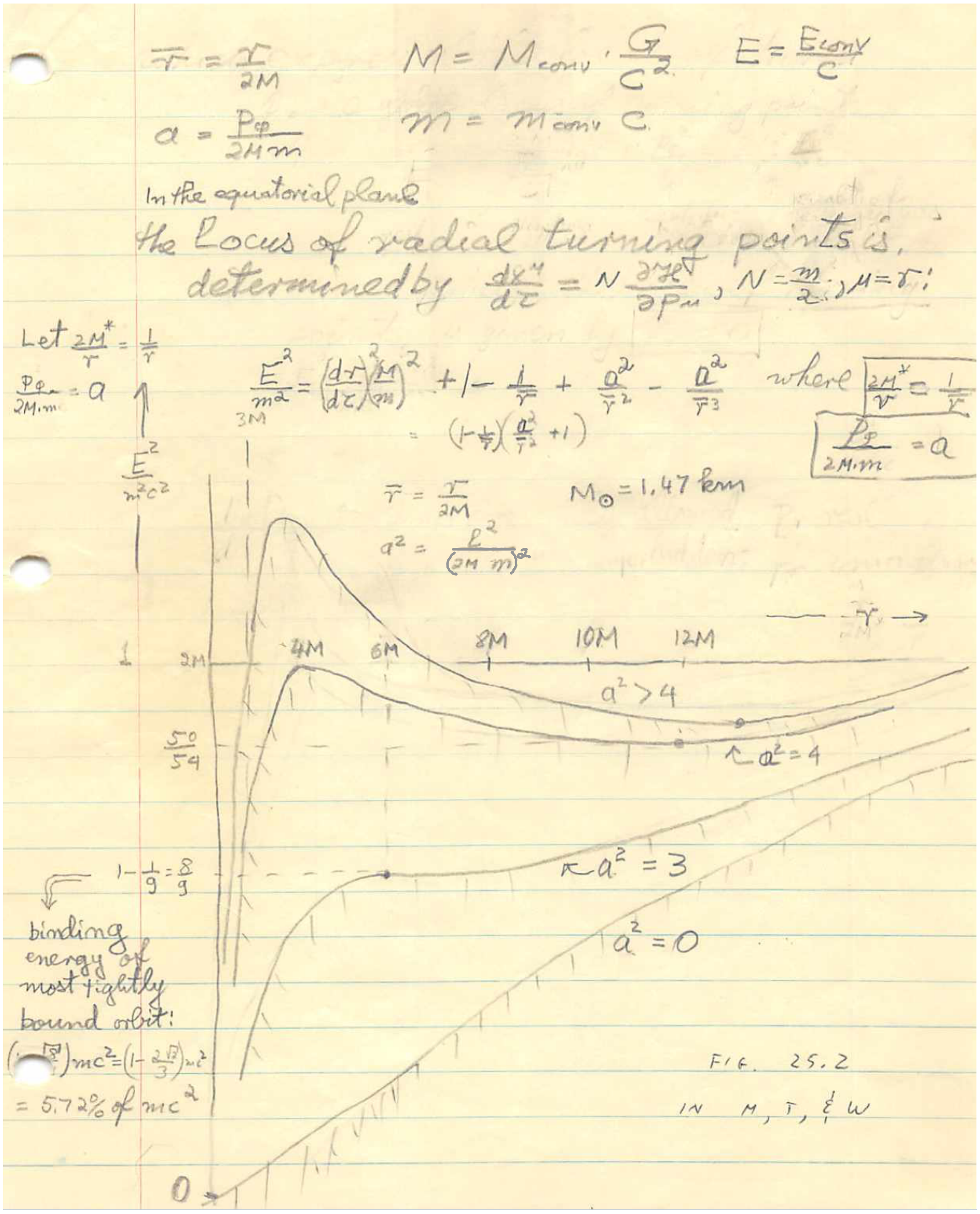
Upon introducing dimensionless quantities

$$\frac{2M}{r}=\frac{1}{\bar{r}}\,,\quad \frac{p^2}{2Mm}=a$$

we obtain the following contributions to the radial potential:

$$\frac{E^2}{m^2}=\underbrace{1}_{\substack{\text{rest}\\\text{mass}}}-\underbrace{\frac{1}{\bar{r}}}_{\substack{\text{Newtonian}\\\text{attraction}}}+\underbrace{\frac{a^2}{\bar{r}^2}}_{\substack{\text{centrifugal}\\\text{repulsion}}}-\underbrace{\frac{a^2}{\bar{r}^3}}_{\substack{\text{Angular}\\\text{kinetic}\\\text{energy has}\\\text{weight.}}}=\left(1-\frac{1}{\bar{r}}\right)\left(1+\frac{a^2}{\bar{r}^2}\right)$$

which expresses the locus of turning points that seperates a classically allowed from a classically forbidden region.



$\bar{r} = \frac{r}{2M}$ $M = M_{conv} \cdot \frac{G}{c^2}$ $E = \frac{E_{conv}}{c}$

$a = \frac{p_\varphi}{2Mm}$ $m = m_{conv}\, c$

In the equatorial plane

the locus of radial turning points is, determined by $\frac{dx^\mu}{d\tau} = N \frac{\partial \mathcal{H}}{\partial p_\mu}$, $N = \frac{m}{2}$, $\mu = r$:

Let $\frac{2M^*}{r} = \frac{1}{\bar{r}}$

$\frac{p_\varphi}{2M \cdot m} = a$

$$\frac{E^2}{m^2} = \left(\frac{d\bar{r}}{d\tau}\right)^2 \left(\frac{M}{m}\right)^2 + 1 - \frac{1}{\bar{r}} + \frac{a^2}{\bar{r}^2} - \frac{a^2}{\bar{r}^3} \quad \text{where } \boxed{\frac{2M^*}{r} = \frac{1}{\bar{r}}}$$

$$= \left(1 - \frac{1}{\bar{r}}\right)\left(\frac{a^2}{\bar{r}^2} + 1\right)$$

$\boxed{\frac{p_\varphi}{2M \cdot m} = a}$

$\bar{r} = \frac{r}{2M}$ $M_\odot = 1.47$ km

$a^2 = \frac{\ell^2}{(2M\,m)^2}$

binding energy of most tightly bound orbit:

$\left(1 - \frac{\sqrt{8}}{3}\right) mc^2 = \left(1 - \frac{2\sqrt{2}}{3}\right) mc^2$

$= 5.72\%$ of mc^2

FIG. 25.2

IN M, T, & W

Figure 39.1 Locus of turning pts of particles have angular momenta $p_\varphi = 0, \sqrt{3}(2Mm), 2(2Mm); p_\varphi > 2(2Mm)$.

We note that for large enough angular momentum ($p_\varphi > \sqrt{3}\, 2M\cdot m$)

(i) there is bounded motion $E < m$

unbounded motion $E > m$

as well as motion in which the particle disappears into the black hole ($r < 2M$)

(ii) there exist stable ("Newtonian") as well as unstable ("relativistic") circular orbits.

They are determined by

$\frac{dE^2(r)}{dr} = 0$, which implies

$$\frac{r}{2M} = a^2\left(1 \pm \sqrt{1 - \frac{3}{a^2}}\right) \qquad a = \frac{p_\varphi}{2Mm}$$

From the catalogue of circular orbits

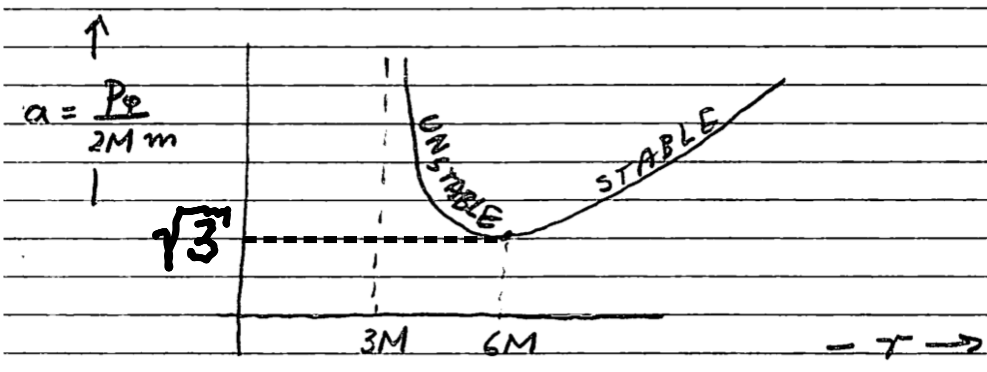


Figure 39.2 Circular orbits of particles catalogued by their angular momentum p_φ.

one can see that there exist no circular orbits, stable or unstable, for $r < 3M$.

and that the most tightly bound stable circular orbit has radius

$r = 6M$,