LECTURE 4

Geometrization of Newtonian Mechanics

I. Free Particle in a Rotating Frame

II. Free Body Motion vs. Geodesic Motion

III. Free Body Motion in an Accelerated Frame

IV. Geometrization of Gravitation } consigned to Lecture 5

In MTW read the caption to Figure 1.7

In H. Goldstein's "Classical Mechanics" read Rate of change of a vector, which is
Section 4.8-4.9 in the 1st (1959) edition, pages 132-135
Section 4.9-4.10 in the 3rd (2000) edition, pages 171-176
(with C. Poole and J. Safko as coauthors) [PDF copy is available over the internet]

(4.1)

The process of geometrizing (= putting into geometrical form) gravity is an inductive process. It asks and answers the question "Why?" This question is at the center of the law of causality*.

* \footnote{This fundamental law is concretized, identified, defined, and characterized in L. Peikoff's "OBJECTVISM: THE PHILOSOPHY OF AYN RAND" pages 12-17.}

There are several ingredients in that inductive process. Two of them are the motions of a free particle as observed

(i) in a <u>rotating</u> frame
(ii) in an <u>accelerated</u> frame

I. Motion of Bodies Relative to an Inertial Frame

(4.2)

However, the point of departure for this process is an inertial (= "free float") frame. It is defined by Newton's 1st Law of motion:

Bodies in uniform motion remain in their states of constant velocities along straight lines, unless their states are

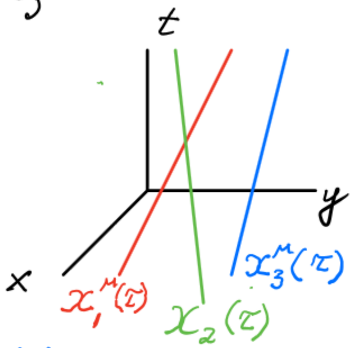

Figure 4.1 In an inertial reference frame all free particle motions are observed to be straight lines. Newton's 1st Law of motion is used to identify an inertial frame by this observed fact.

are changed by forces impelled on them.

II. Motion of Bodies Relative to a Rotating Frame

(4.3)

Q: What is the motion of each of these bodies as observed relative to a rotating frame of reference?

A: The observed motion is non-uniform in that it is characterized by Coriolis and centrifugal acceleration; from the geometrical perspective their components are <u>those</u> Christoffel symbols Γ^i_{jk} which are <u>non-zero</u>.

Indeed, consider a rigid frame of reference which rotates with angular velocity $\vec{\omega}$ <u>relative to the fixed stars</u>. A frame is said to be <u>rigid</u> if all of its basis vectors rotate around an axis with the same angular velocity. A vector is said to be <u>attached rigidly</u> to such a frame if its components relative to the frame's basis are independent of time.

Consider a vector, say $\vec{G}$, which is rigidly attached to this frame. This vector will rotate relative to the static inertial

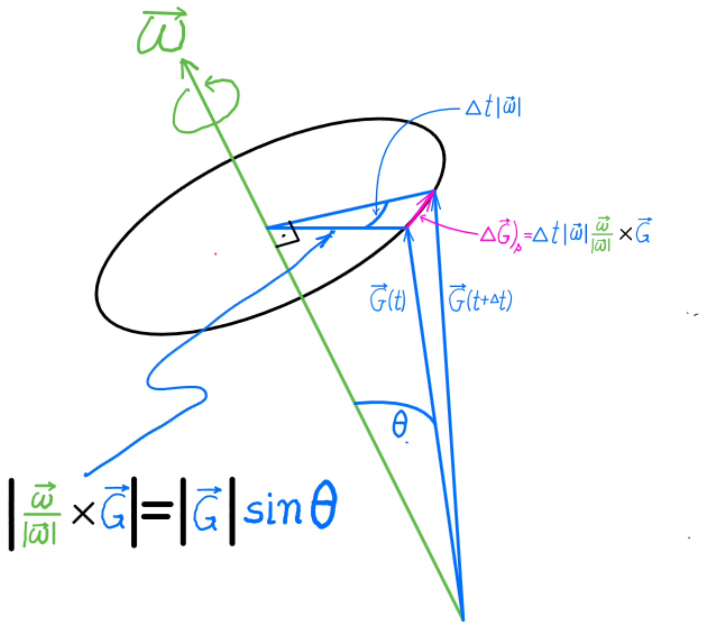

Figure 4.2 Location vector $\vec{G}(t)$ rotates around the rotation axis $\frac{\vec{\omega}}{|\vec{\omega}|}$ at angular rate $|\vec{\omega}|$. During Δt the rotation of $\vec{G}$ is expressed by the vectorial displacement $\Delta\vec{G}$. This vector is

$$\Delta\vec{G} = \Delta t\,\vec{\omega}\times\vec{G}.$$

frame, namely relative to the fixed stars. As depicted is Figure 4.2, during a time interval Δt this rotating vector

$\vec{G}$ will have changed by the amount (4.5)

$$\Delta\vec{G})_s = \Delta t \; \vec{\omega}\times\vec{G},$$

which is perpendicular to both $\vec{\omega}$ and $\vec{G}$. The vector $\Delta\vec{G})_s$ expresses an infinitesimal change (relative to the fixed stars) in $\vec{G}$ due to an infinitesimal rotation by an amout $\Delta t\,|\vec{\omega}|$ around the direction $\vec{\omega}$.

However, consider the circumstance where the vector $\vec{G}$ is <u>not rigidly</u> attached to the rotating frame. But instead changes by the amount $\Delta\vec{G})_{rot}$ in the rotating frame during the time interval Δt. In this case the relation between the change $\Delta\vec{G})_s$ relative to the static ("fixed stars") frame and the change $\Delta\vec{G})_{rot}$ relative to the rotating frame is

(4.6)

$$\Delta\vec{G}\Big)_s = \Delta\vec{G}\Big)_{rot} + \Delta t\, \vec{\omega}\times G$$

Thus

$$\boxed{\frac{d\vec{G}}{dt}\Big)_s = \frac{d\vec{G}}{dt}\Big)_{rot} + \vec{\omega}\times\vec{G}} \qquad (4.1)$$

Apply this kinematic relation to the position vector $\vec{R}(t)$:

$$\frac{d\vec{R}}{dt}\Big)_s = \frac{d\vec{R}}{dt}\Big)_{rot} + \vec{\omega}\times\vec{R},$$

and then again to the velocity vector $\vec{v}_s = \frac{d\vec{R}}{dt}\Big)_s$.

One finds that for a free particle

$$0 = \frac{d}{dt}\left[\frac{d\vec{R}}{dt}\Big)_s\right]\Bigg)_s = \frac{d}{dt}\left[\frac{d\vec{R}}{dt}\Big)_{rot} + \vec{\omega}\times\vec{R}\right]\Bigg)_{rot} + \vec{\omega}\times\left[\frac{d\vec{R}}{dt}\Big)_{rot} + \omega\times\vec{R}\right] \qquad (4.2)$$

$$\frac{d}{dt}\left[\frac{d\vec{R}}{dt}\Big)_{rot}\right]_{rot} = \vec{a}_{rot} = \frac{d^2x^i}{dt^2}\,e^*_i(t)$$

$$\frac{d\vec{R}}{dt}\Big)_{rot} = \vec{v}_{rot} = \frac{dx^i}{dt}\,e^*_i(t)$$

rotating frame basis vectors

Our focus of attention is a rotating (4.7)
frame of CONSTANT time-independent angular velocity $\vec{\omega}$.

Thus for a freely moving body of mass m, its acceleration and hence the forces impelling it as observed relative to a rotating frame is

$$\left\{ m\frac{d^2x^i}{dt^2} = \underbrace{-m2\left[\vec{\omega}\times\vec{v}_{rot}\right]^i}_{\text{Coriolis force}} \underbrace{-m\left[\vec{\omega}\times(\vec{\omega}\times\vec{R})\right]^i}_{\text{Centrifugal force}} \right\} e_i^*(t) \qquad (4.3)$$

$$\text{Centrifugal force} = -m(\vec{\omega}\cdot\vec{R})\vec{\omega} + m\,\omega^2\vec{R}$$

\footnote{ The "Centrifugal force" is called that because $\vec{\omega}\times(\vec{\omega}\times\vec{R})$ points perpendicularly away from the axis of rotation and has magnitude $\vec{\omega}^2 R \sin\theta$. This is simply $\vec{\omega}^2\cdot$(radius of the tip of $\vec{R}=\vec{G}$ away from the rotation axis) in Figure 4.2.

The Coriolis and the centrifugal forces are inertial pseudo forces. "Inertial" because they are proportional to the mass. "Pseudo" because of the non-inertial nature of that rotating frame.

(4.8)

III. Free Body vs. Geodesic Motion

Compare the 1st component (relative to the rotation basis $\{e^*_i\}$) of the free particle motion with the $\mu=1$ component of a geodesic

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}=0\,, \qquad (4.4)$$

namely,

$$0=\frac{d^2x^1}{dt^2}+2\left(\omega_2\frac{dx^3}{dt}-\omega_3\frac{dx^2}{dt}\right)+\vec{\omega}\cdot\vec{R}\,\omega_1-\vec{\omega}\cdot\vec{\omega}\,x^1 \qquad (4.5)$$

with

$$0=\frac{d^2x^1}{d\tau^2}+2\,\Gamma^1_{0k}\frac{dt}{d\tau}\frac{dx^k}{d\tau}+\Gamma^1_{00}\frac{dt}{d\tau}\frac{dt}{d\tau}+\Gamma^1_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} \qquad (4.6)$$

In the non-relativistic approximation, the body's proper time τ equals the time t in the rotating frame:

$$\frac{dt}{d\tau}=1.$$

It is understood that time t is measured in units of distance travelled by light.

This implies that

$$t = c\,t_{conventional}$$

$$\vec{\omega} = \frac{\vec{\omega}_{conventional}}{c}$$

and that in the non-relativistic limit

$$\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = \frac{1}{c^2}\frac{dx^j}{dt_{conv}}\frac{dx^k}{dt_{conv}} \ll 1,$$

which therefore is neglegibly small in Eq. (4.6).

In the asymptotic non-relativic limit, with an appropriate choice of the Γ's, the differential Eq. (4.6) is the same as Eq. (4.5) for all functions $x^1(t)$, $x^2(t)$, and $x^3(t)$. The same observations apply to Eq. (4.3) and (4.4) when $i = 2, 3$ and $\mu = 2, 3$ respectively.

Consequently, the Γ-terms in Eq. (4.6) are related to those in Eq. (4.5) by the following equations:

a) $$\begin{bmatrix} \Gamma^1_{01} & \Gamma^1_{02} & \Gamma^1_{03} \\ \Gamma^2_{01} & \Gamma^2_{02} & \Gamma^2_{03} \\ \Gamma^3_{01} & \Gamma^3_{02} & \Gamma^3_{03} \end{bmatrix} \begin{bmatrix} \frac{dx^1}{d\tau} \\ \frac{dx^2}{d\tau} \\ \frac{dx^3}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} \frac{dx^1}{d\tau} \\ \frac{dx^2}{d\tau} \\ \frac{dx^3}{d\tau} \end{bmatrix}$$

or $$\boxed{\Gamma^{i}{}_{0k} = \epsilon_{ijk}\,\omega^{j}} \qquad \left.\begin{matrix} i \\ k \end{matrix}\right\} = 1,2,3 \tag{4.10}$$

b) $$\boxed{\Gamma^{i}{}_{00}\underbrace{\frac{dt}{d\tau}}_{1}\underbrace{\frac{dt}{d\tau}}_{1} = \vec{\omega}\cdot\vec{R}\,\omega_i - \vec{\omega}\cdot\vec{\omega}\,x^{i}} \qquad i = 1,2,3$$

c) $$\boxed{\Gamma^{i}{}_{jk}\frac{dx^{j}}{dt}\frac{dx^{k}}{dt} = \Gamma^{i}{}_{jk}\frac{dx^{j}}{dt_{conv}}\frac{dx^{k}}{dt_{conv}}\frac{1}{c^2} \approx 0}$$

non-rel. approx'n

IV. Conclusions:

a) Coriolis acceleration $\neq 0 \Rightarrow \Gamma^{i}{}_{0k} \neq 0$

b) Centrifugal acceleration $\neq 0 \Rightarrow \Gamma^{i}{}_{00} \neq 0$

c) In non-relativistic mechanics $\left.\frac{dx^{j}}{dt_{conv}}\frac{dx^{k}}{dt_{conv}}\frac{1}{c^2} \ll 1\right\} \Rightarrow \Gamma^{i}{}_{jk}\frac{dx^{j}}{dt}\frac{dx^{k}}{dt} \approx 0$