Lecture 40

Schwarzschild Spacetime:
It's topological and
causal structure

In MTW read §31.6, Figure 31.5 } Topological structure
" " do Exercise 31.7
" " read Box 31.2 } Causal structure
" " read Section 32.3

} For a Spherical Vacuum Configuration

(40.1)

I. The Schwarzschild Spacetime

In 1923 G. Birkoff showed that

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \qquad (40.1)$$

which was first exhibited by K. Schwarzschild, is the only solution to the E.F.E.s in vacuum which is spherically symmetric. The Schwarzschild coordinates relative to which the Schwarzschild solution is a mathematical package deal: a combination of good and non-good; it directs attention to the spatial topological structure while at the same time lacking information about its causal structure. The latter is exposed relative the Eddington-Finkelstein coordinates. But to exhibit both structures requires the representation by means of the globally defined Kruskal-Szekeres coordinates.

A. Schwarzschild Coordintes: Spatial Topological Structure.

The most eye-catching aspect of the Schwarzschild

(40.2)

metric, Eq. (40.1), is its mathematical behavior at $r=2M$. This, however, is not signal some sort of extreme physical behavior.

1. Indeed, consider using a plumb line to measure the proper distance from some fixed radius ($r=\sqrt{\frac{\text{area}}{4\pi}}$) down to $r=2M$: $\displaystyle\int_{r}^{r=2M}\frac{dr'}{\sqrt{1-\frac{2M}{r'}}} = \text{finite}.$

2. Second, consider the time of proper time it takes for a particle to plunge from some finite radius down tor $r=2M$. From Eq. (39.11) one has

$$\tau = \int_{r}^{r=2M}\frac{dr'}{\left\{E^2-\left(\frac{L^2}{r'^2}+m^2\right)\left(1-\frac{2M}{r'}\right)\right\}^{1/2}} = \text{finite}$$

3. Third consider the physical (O.N.) curvature components

$$\hat{R}_{\alpha\beta\gamma\delta} = \left\{\begin{matrix} \pm\frac{2M}{r} \\ \pm\frac{M}{r}\end{matrix}\right\} = \text{finite} \quad \text{(Relive to the Schsch. frame)}$$

$$\hat{R}_{\alpha\beta\gamma\delta} = \left\{\begin{matrix} \pm\frac{2M}{r} \\ \pm\frac{M}{r}\end{matrix}\right\} = \text{finite} \quad \left(\begin{matrix}\text{Relative to any frame} \\ \text{with arbitrary radial} \\ \text{velocity relative to the} \\ \text{Schsch frame.}\end{matrix}\right)$$

40.3

4. The imbedding diagram of the equatorial plane $\theta = \frac{\pi}{2}$ at $t = \text{const}$ in a 3-d imbedding space

for
$$d\sigma^2 = \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\varphi^2$$
$$= dz^2 + dr^2 + r^2 d\varphi^2$$

yields
$$\left(\frac{dz}{dr}\right)^2 + 1 = \frac{1}{1-\frac{2M}{r}} \Rightarrow z = \pm\sqrt{8M(r-2M)}$$

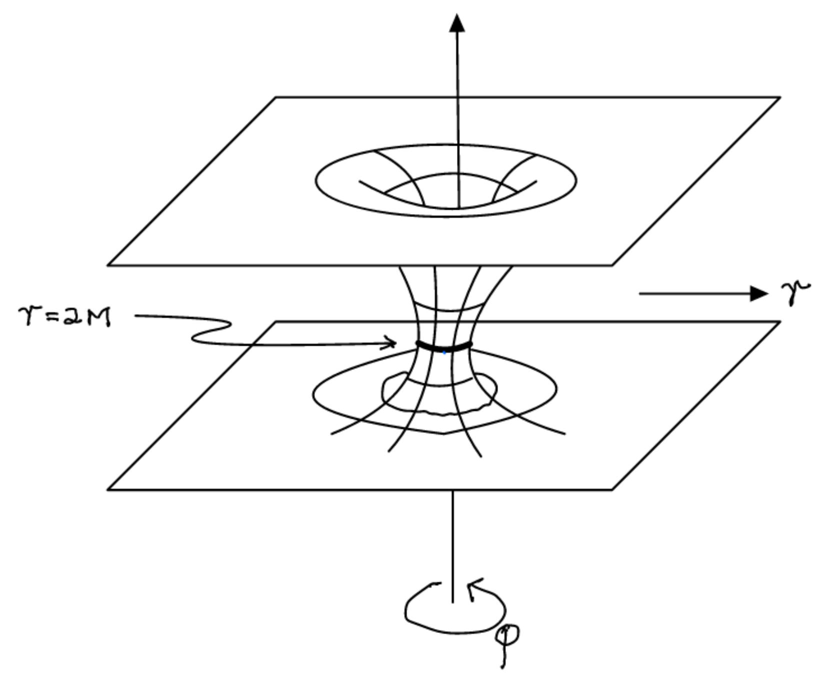


Figure 40.1 The topology of the spatial (Schsch.) equatorial plane is that of two asymptotically flat domains connected by a throat whose minimal radius is 2M.

The two asymptotically flat 3-d spaces are isometric (40.4) copies of each other. Indeed, this isometry becomes evident when one changes from the radial Schsch coordinate to the radial conformal coordinate,

$$r \to \rho,$$

as determined by the condition that

$$d\sigma^2 = -\frac{dr^2}{1-\frac{2M}{r}} + r^2 d\varphi^2 = f^2(\rho)\left[d\rho^2 + \rho^2 d\varphi^2\right].$$

By equating coefficients one obtains

$$\left.\begin{aligned} r &= \rho f(\rho) \\ \frac{1}{\sqrt{1-\frac{2M}{r}}} &= f(\rho)\frac{d\rho}{dr} \end{aligned}\right\} \frac{dr}{d\rho} = \frac{r}{\rho}\sqrt{1-\frac{2M}{r}}$$

Consequently,

$$r = \rho\left(1+\frac{M}{2\rho}\right)^2 \text{ and } f(\rho) = \left(1+\frac{M}{2\rho}\right)^2.$$

(i) The graph of the Schsch radial coordinate

$$r \equiv \frac{\text{circumference}}{2\pi}$$

in terms of the conformal coordinate ρ is depicted in Figure 40.2

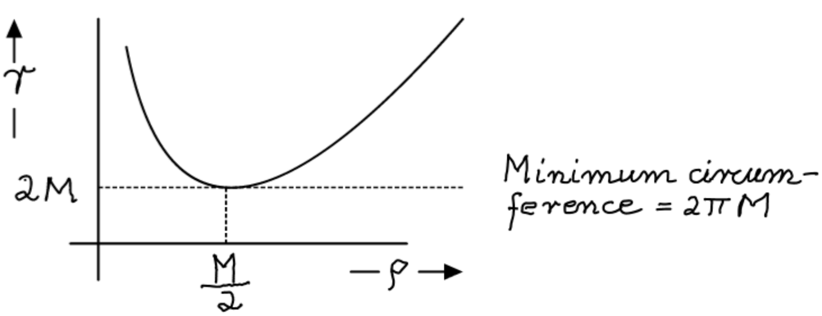


Figure 40.2

(ii) Relative to this conformal radial coordinate (40.5)

the Schsch metric is

$$ds^2 = -\frac{\left(1-\frac{M}{2\rho}\right)^2}{\left(1+\frac{M}{2\rho}\right)^2}dt^2 + \left(1+\frac{M}{2\rho}\right)^4\left[d\rho^2+\rho^2(d\theta^2+\sin^2\theta\, d\varphi^2)\right] \qquad (40.2)$$

(iii) This geometry is symmetric around $\rho = \frac{M}{2}$

under the <u>isometric</u> transformation

$$\frac{2\rho}{M} \longrightarrow \frac{M}{2\bar\rho} \qquad \left(\rho = \left(\frac{M}{2}\right)^2\frac{1}{\bar\rho}\right) \qquad (40.3)$$

This transformation maps the entire manifold $\rho > 0$ onto itself with the same metric:

$$ds^2 = -\frac{\left(1-\frac{M}{2\bar\rho}\right)^2}{\left(1+\frac{M}{2\bar\rho}\right)^2}dt^2 + \left(1+\frac{M}{2\bar\rho}\right)^4\left[d\bar\rho^2+\bar\rho^2(d\theta^2+\sin^2\theta\, d\varphi^2)\right]$$

(iv) Conclusion:

The global static Schsch solution to the EFEs is a mathematization of two asymptotically flat isometric 3-d spaces depicted in Figure 40.1.

B. Causal Structure: The Eddington-Finkelstein coordinatization (40.6)

The time invariant representation of the Schsch metric, Eq. (40.1) on page 40.1, is deficient. This is because the $r=2M$ singularity in the time component of the metric hides the causal structure of Schsch spacetime.

Indeed, consider the causal structure of radial light cones spanned by (t, r) at $\theta = \text{const}$, $\varphi = \text{const}$ on $M^2 = M^4/S^2$:

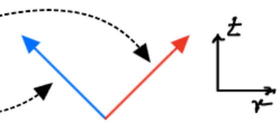
$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}}.$$

The light cones are generated by tangents to the photon world lines (= "null rays")

$(\Delta s)^2 = 0$:

$$\frac{dr}{dt} = \left(1-\frac{2M}{r}\right) \quad \text{outgoing} \qquad (40.4)$$

$$\frac{dr}{dt} = -\left(1-\frac{2M}{r}\right) \quad \text{ingoing} \qquad (40.5)$$



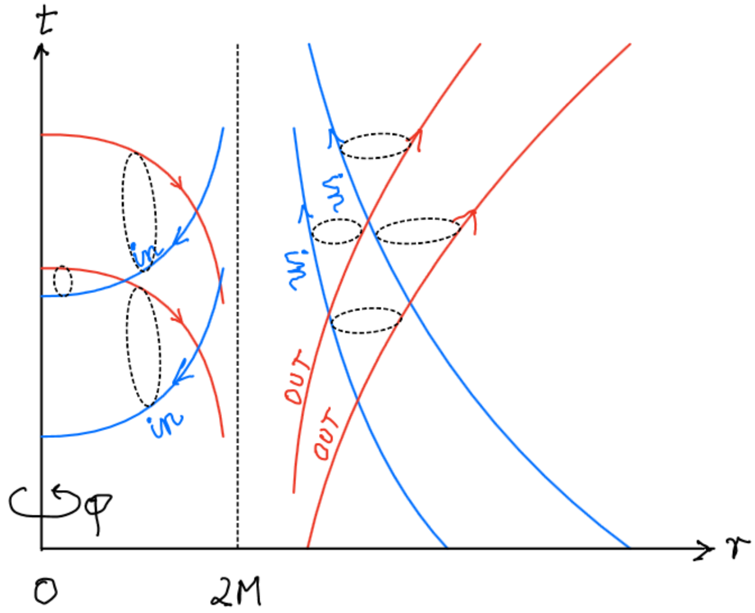


Figure 40.3 Ingoing and outgoing world lines of classical photons

The metric $r=2M$ singularity exposes the fact that t-coordinate is a bad coordinate at $r=2M$.

This is because there are many ingoing geodesics all of them governed by

$$\frac{dr}{dt} = -\left(1-\frac{2M}{r}\right). \qquad (40.5)$$

They all converge to $t=\infty$, $r=2M$,

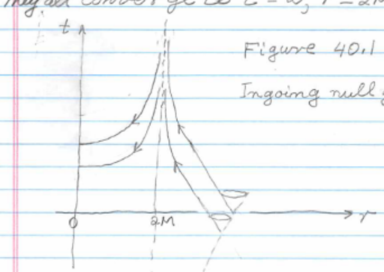


Figure 40.1 Ingoing null geodesics

Figure 40.4

and the diff'l Eq. (40.5) does not determine which ingoing null geodesic outside, i.e.

(40.8)

$2M<r$, goes with which ingoing null geodesic inside, i.e. $r<2M$.

Furthermore, is ($t=\infty$, $r=2M$) a single event or is a set of distinct events?

The same ambiguity holds for outgoing radial null geodesics which are mathematized by the d.e.

$$\frac{dr}{dt}=\left(1-\frac{2M}{r}\right).$$

They all diverge from ($t=-\infty$, $r=2M$)

Figure 40.5

These ambiguities have been resolved by Eddington (1924) and Finkelstein (1958). They integrated Eq. (40.5) and introduced the integration constant as a new coordinate function that replaces the "bad" Schsch time coordinate t: The integrals of

$$0=dt+\frac{dr}{1-\frac{2M}{r}}\equiv dt+dr^*$$

are

$$\tilde{V}=t+r+2M\ln\left(\frac{r}{2M}-1\right)=\tilde{V}$$

40.9

The isograms of

$$\tilde{V}(t,r) = t + r + 2M \ln\left(\frac{r}{2M} - 1\right) = \text{const.}$$
$$= t + r^* = \text{const} \qquad (40.6)$$

are the ingoing null geodesics:

$\tilde{V}$ is their new coordinate function. It called the advanced

Schsch time coordinate,

Similarly, the isograms of

$$\tilde{U}(t,r) = t - r^* = \text{const} \qquad (40.7)$$

are the outgoing null geodesics;

and $\tilde{U}$ is called the retarded

Sch sch time coordinate.

The radial variable

$$r^* \equiv r + 2M \ln\left(\frac{r}{2M} - 1\right) \qquad (40.8)$$

is called the "tortoise coordinate".

Introduce $(\tilde{V}, r)$ as the (new) ingoing Eddington-Finkelstein coordinates for the Sch sch geometry:

$$\left.\begin{aligned} d\tilde{V} &= dt + \frac{dr}{1-\frac{2M}{r}} \\ dr &= dr \end{aligned}\right\} \quad dt = d\tilde{V} - \frac{dr}{1-\frac{2M}{r}} \qquad (40.9)$$

Relative to these new coordinates

the Schsch metric becomes non-diagonal,

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega^2 \qquad (40.10)$$

$$= -\left(1-\frac{2M}{r}\right)\left(dt + \frac{dr}{1-\frac{2M}{r}}\right)\left(dt - \frac{dr}{1-\frac{2M}{r}}\right) + {}''$$

$$= -\left(1-\frac{2M}{r}\right) d\tilde{V}\left(d\tilde{V} - \frac{2\,dr}{1-\frac{2M}{r}}\right) + {}''$$

$$\boxed{ds^2 = -\left(1-\frac{2M}{r}\right)d\tilde{V}^2 + 2\,dV\,dr + r^2 d\Omega^2}, \qquad (40.10)$$

but it is nonsingular at $r=2M$.

Thus ingoing radial null geodesics are characterized by

$$\frac{dt}{dr} + \frac{1}{1-\frac{2M}{r}} = \boxed{\frac{d\tilde{V}}{dr} = 0} \qquad (40.11)$$

while outgoing radial null geodesics are characterized by

$$0 = \frac{dt}{dr} - \frac{1}{1-\frac{2M}{r}} = \frac{d\tilde{V}}{dr} - \frac{2}{1-\frac{2M}{r}}$$

$$\text{or} \quad \boxed{\frac{d\tilde{V}}{dr} = \frac{2}{1-\frac{2M}{r}}} \qquad (40.12)$$

The two boxed Eqs. (40.11) - (40.12) are the tangents to the two sets of null geodesics

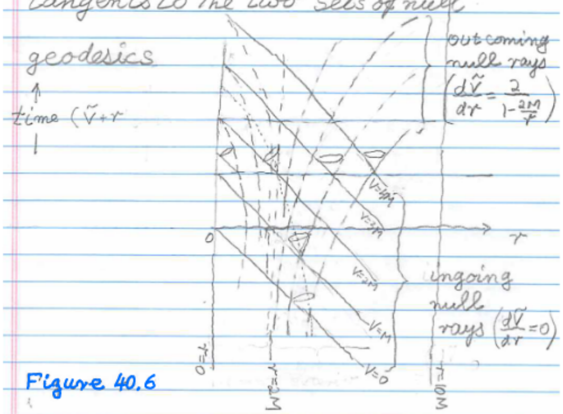


Figure 40.6

Ingoing Eddington-Finkelstein coordinate system $(\tilde{V}, r)$.

(40,11)

b) Outgoing E-F ("Retarded time") coordinates.

These coordinates are based on outgoing null geodesics

$$\frac{dr}{dt} = 1 - \frac{2M}{r},$$

the integrals of

$$0 = dt - \underbrace{\frac{dr}{1-\frac{2M}{r}}}_{dr^*}$$

are

$$\tilde{U} = t - \overbrace{\left(r + 2M \ln\left(\frac{r}{2M} - 1\right)\right)}^{r^*}$$

The isograms of

$$\tilde{U}(t,r) = t - \left(r + 2M \ln\left(\frac{r}{2M} - 1\right)\right)$$

are the outgoing null geodesics, and $\tilde{U}$ is called the retarded Schsch. time coordinate.

40,12

Relative to the $(\tilde{U}, r)$ coordinate system the Schsch metric has the form

$$ds^2 = -\left(1-\frac{2M}{r}\right) d\tilde{U}^2 - 2\,d\tilde{U}\,dr + r^2 d\Omega^2$$

The outgoing null geodesics obey

$$\frac{d\tilde{U}}{dr} = 0$$

while the ingoing null geodesics obey

$$\frac{d\tilde{U}}{dr} = \frac{-2}{1-\frac{2M}{r}}$$

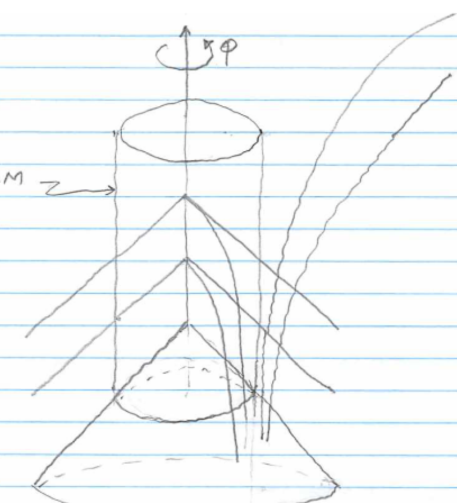


Figure 40.7
INGOING Eddington-Finkelstein coordinates system $(\tilde{V}, r)$.