Lecture 41

Spacetime geometry of a spherical black hole and "white hole"

In MTW peruse Box31.2

Inspite of the fact that the black hole metric representation

$$ds^2 = -\left(1-\frac{2M}{r}\right)\left[dt^2 - \frac{dr^2}{\left(1-\frac{2M}{r}\right)^2}\right] + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \qquad (41.1)$$

exhibits an obvious pathology at $r=2M$, physical and geometrical scrutiny indicate that the opposite is the case (finite proper distances, proper times, curvature, smoothness of the spatial geometry): $r=2M$ is <u>not</u> a physical singularity.

The cause and the cure of this mathematical pathology is found once one takes cognizance of the central role of radial light pulses in representing the black hole metric.

The spacetime trajectories (a.k.a. "null rays" or "null geodesics") of these light pulses are solutions to the differential equations obtained by inspecting Eq.(41.1),

$$dt^2 - \frac{dr^2}{\left(1-\frac{2M}{r}\right)^2} = 0 \Rightarrow \begin{cases} dt + \dfrac{dr}{1-\frac{2M}{r}} \equiv d\tilde{V} = 0 \longrightarrow t + r + 2M \ln\left(\frac{r}{2M}-1\right) = \tilde{V} & (41.2) \\[2ex] dt - \dfrac{dr}{1-\frac{2M}{r}} \equiv d\tilde{U} = 0 \longrightarrow t - r - 2M \ln\left(\frac{r}{2M}-1\right) = \tilde{U} & (41.3) \end{cases}$$

It is a fact that for $r>2M$ an observer distinguishes between ingoing and outging light pulse world lines. They are mathematized by Eqs. (41.2) and (41.3) and their depiction relative to Schsch coordinates are given in Figures 41.1 and 41.3 (41.2)

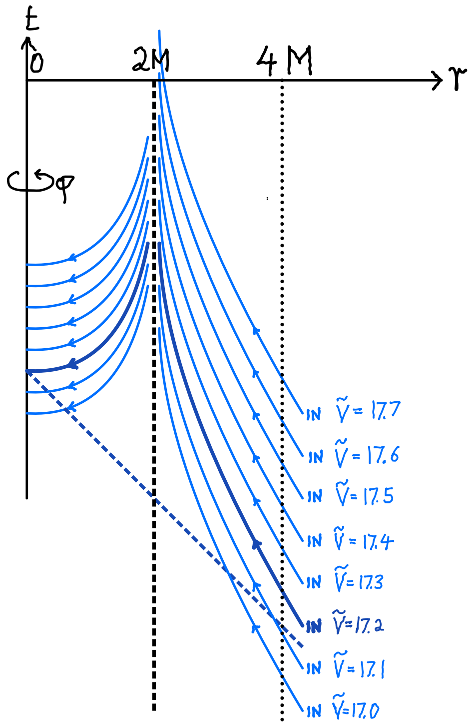


Figure 41.1 World lines of ingoing light pulses travelling towards $r=2M$ of a black hole. Every such world line **has its own value of $\tilde{V}$ as a constant of motion.** Equation (41.2) necessitates that the Schsch slope of every world line depicted in the figure is

$$\frac{dt}{dr} = -\frac{1}{1-\frac{2M}{r}},$$

and that their graphs in the Schsch plane are

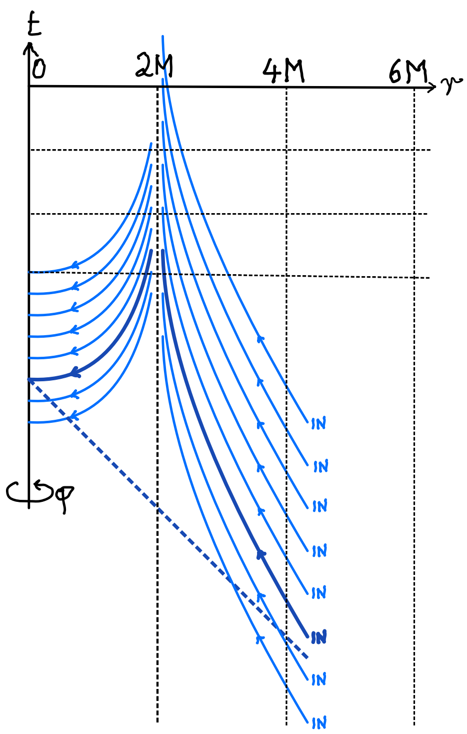
$$t + r + 2M \ln\left|\frac{r}{2M} - 1\right| = \tilde{V}.$$

The exterior domain of a black hole is $r > 2M$. Its interior is $r < 2M$.

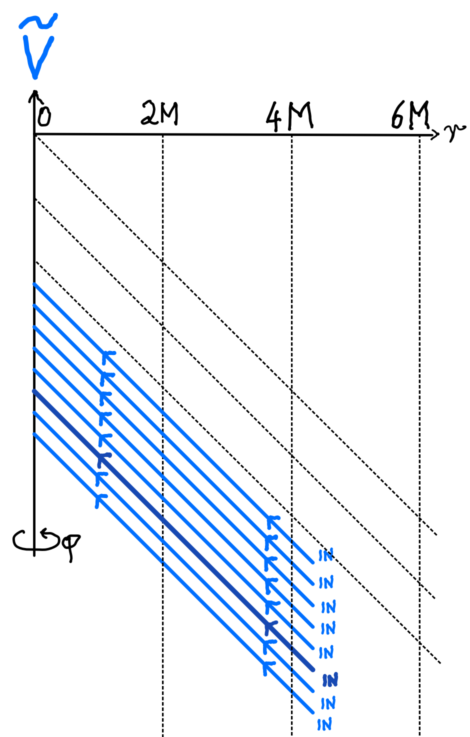
But this partitioning, with the $r = 2M$ boundary between the two, and knowing which event on one side is close to an event on the other, requires a representation of the black hole metric which is continuous across both domains. Transitioning from the Schsch to the <u>ingoing</u> Eddington-Finkelstein coordinates fulfills this requirement. As shown in the text, the black hole metric is indeed continuous across both domains. As a consequence each ingoing light pulse world continues it evolution uniquely into the interior $r < 2M$. The heavy dark trajectory in the interior is the continuous evolution of the ingoing one in the exterior:

Even though $r=2M$ is not a physical singularity (i.e. all measurable attributes involving $r=2M$ are finite and well-defined), the Schsch coordinate representation of the spacetime metric of a black hole is singular at $r=2M$. According to this mathematical representation, no ingoing entity, including light pulses, can pass through $r=2M$. This contradiction is removed by changing to a coordinate system relative to which the black hole metric is non-singular at $r=2M$.

The coordinate transformation $(t,r)\rightarrow(\tilde{V},r)$ of Eq. (41.2)



A



B

Figure 41.2 Standard (A) and light pulse-induced (B) coordinatizations of the exterior and the interior of a black hole.

Panel A: World lines of ingoing light pulses relative to Schsch cordinates (t, r).

Panel B: World lines of ingoing light pulses are straightened out by the ingoing Eddington-Finkelstein coordinates $(\tilde{V}, r)$. They terminate mathematically at $r=0$.

yields a representation of the black hole metric which is non-singular across $r=2M$,

$$ds^2 = -\left(1-\frac{2M}{r}\right) d\tilde{V}\left(d\tilde{V} - 2\frac{dr}{1-\frac{2M}{r}}\right) + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$= -\left(1-\frac{2M}{r}\right) d\tilde{V}^2 + 2\, d\tilde{V}\, dr + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

The world lines of radial ingoing light pulses trace out the isograms of the coordinate function $\tilde{V}$ of the oblique $\tilde{V}$-r coordinate system. They terminate mathematically at $r=0$.

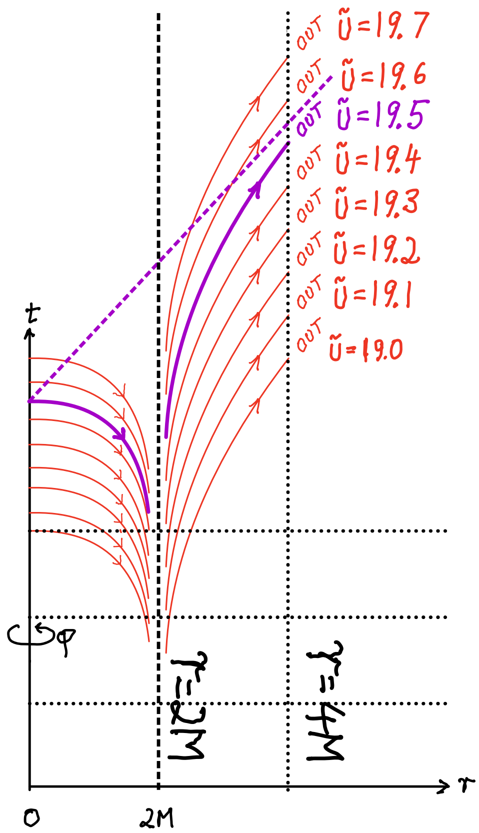


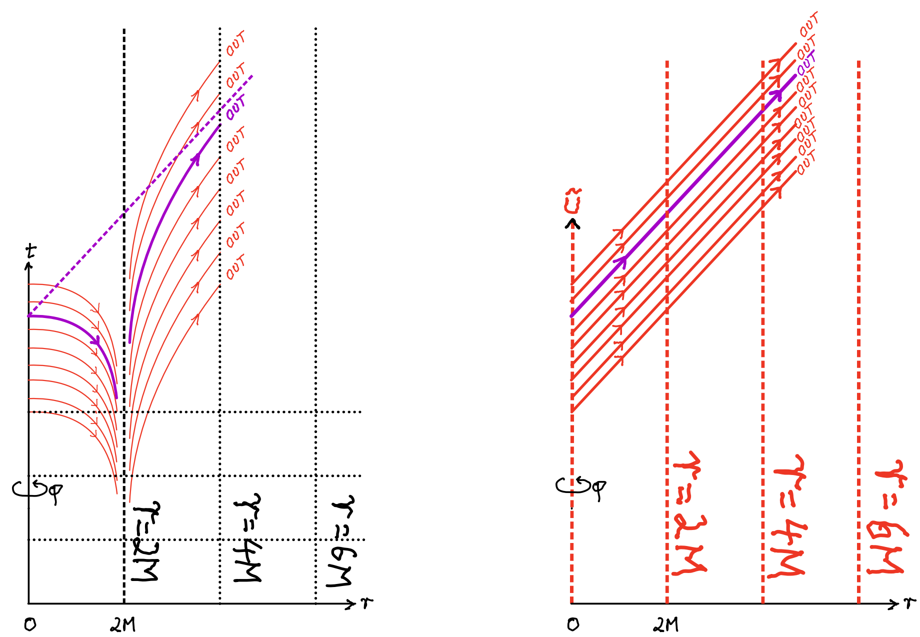
Figure 41.3 World lines of outgoing light pulses coming from $r=2M$ of a "white hole". Every such world line **has its own value of** $\tilde{U}$ **as a constant of motion.** Equation (41.3) necessitates that the Schsch slope of every world line depicted in the figure is

$$\frac{dt}{dr}=\frac{1}{1-\frac{2M}{r}},$$

and that their graphs in the Schsch plane are

$$t-r-2M\ln\left|\frac{r}{2M}-1\right|=\tilde{U}.$$

The exterior domain of a "white hole" is $r>2M$. Its interior is $r<2M$.



A. B.

Figure 41.~~x~~4 <---41.4

In the Schsch coordinate representation (panel A) of the "white hole" spacetime, $r=2M$ is not part of the manifold. In any nbhd surrounding $r=2M$ there is no causal connection between radial light pulse worldlines created in $r<2M$ and those in $r>2M$. However, relative to the (outgoing) E-F coordinate representation (panel B), every outgoing radial light pulse in $r>2M$ evolves continuously from a unique light pulse worldline created in $r<2M$.
In particular, the world line depicted by the dark heavy curve in panel B connects (into a single world line) what appears as two dark and heavy disjoint

worldlines relative to the Schsch coordinatized spacetime depicted in panel A.

The representation of the "white hole" metric relative to the outgoing light pulse induced E-F coordinates is

$$ds^2 = -\left(1-\frac{2M}{r}\right)\left[d\tilde{U}\cdot\left(d\tilde{U} + \frac{2}{1-\frac{2M}{r}}\,dr\right)\right] + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$= -\left(1-\frac{2M}{r}\right)d\tilde{U}^2 - 2\,d\tilde{U}\,dr + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

Unlike the Schsch representation, the E-F representation of the metric is non-singular, even at $r = 2M$.