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Physical Foundations of a Theory of Gravitation[1]

by Albert Einstein

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By the word "mass" of a body one denotes two things that are very different according to their definitions: on the one hand, the inertial resistance of the body and, on the other hand, the characteristic constant that is the determining factor for the effect of the gravitational field on the body. It is one of the most remarkable empirical facts of physics that these two masses, the *inertial* and the *gravitational*, agree exactly with each other as regards their magnitude. This agreement was proved most exactly by Eötvös's experiments. A body on the surface of the Earth is acted upon by two generally differently directed forces, which together constitute the apparent gravity of the body: one of these forces, the gravitation proper, depends on the gravitational mass, while the other, the centrifugal force, depends on the inertial mass. By experiments with the torsion balance, Eötvös established that the ratio of these two forces is independent of the nature of the material; in that way he proved the agreement of the two masses of a body with an accuracy that rules out deviations
[2] of the relative magnitude of 10^{-7}.

This empirical law can also be expressed in the following way. In a gravitational field all bodies fall with the same acceleration. This suggests the view that, with regard to its influence on mechanical and other physical processes, a gravitational field may be replaced by a state of acceleration of the reference body (coordinate system). This conception does not follow with necessity from the experiments mentioned, but it is of great heuristic interest all the same. For, since the course of physical processes relative to an accelerated reference system can be determined theoretically, this *equivalence hypothesis* permits us to predict the influence of a
[3] gravitational field on physical processes of every kind. The experimental test of the conclusions so reached must then show whether the underlying hypothesis was correct.

In the way indicated, one comes to the conclusion that the speed with which a physical process occurs in a gravitational field is greater the greater the gravitational potential at the location where the physical system in question is situated. For that reason, the spectral lines of solar light should, for example, experience a small shift toward the red end of the spectrum as compared with the corresponding spectral lines

[1]Based on a lecture delivered on 9 September 1913 at the annual meeting of the
[1] Schweizer Naturforschende Gesellschaft in Frauenfeld.

of terrestrial light sources, namely, a shift of about two millionths of the wavelength.
A further consequence of this equivalence hypothesis is the bending of light rays in [4]
a gravitational field, which amounts to 0.84 seconds of arc for a light ray passing
near the sun and is thus not inaccessible to experimental test. This bending of light [5]
rays implies that the velocity of light is not constant, but depends, instead, on the location. This forces us to generalize the theory of space and time, known as the theory of relativity, since the latter was based on the assumption of the constancy of the velocity of light.

According to the familiar theory of relativity, an isolated material point moves rectilinearly and uniformly according to the equation

$$\delta\left(\int ds\right) = 0,$$

where

$$ds^2 = -dx^2 - dy^2 - dz^2 + c^2dt^2,$$

and c denotes the (constant) velocity of light. The equivalence hypothesis permits the conclusion that in a *static* gravitational field (of special kind) a material point moves according to the above equation, in which now, however, c is a function of location and is determined by the gravitational potential. From this special case of the gravitational field, one can arrive at a general case by passing to moving coordinate
systems by means of coordinate transformation.[2] In this way one recognizes that the [6]
only sufficiently encompassing invariant-theoretical generalization of the indicated law of motion consists in assuming that the "line element ds" has the form

$$ds^2 = \sum_{ik} g_{ik}\,dx_i\,dx_k, \qquad (i,k = 1,2,3,4)$$

where the g_{ik} are functions of x_1, x_2, x_3, and x_4, while the first three coordinates characterize the position, and the last one the time, and the equation of motion is again to have the form

$$\delta\left(\int ds\right) = 0.$$

If one considers that in this view, instead of the customary line element of the original theory of relativity,

$$ds^2 = \sum_i dx_i^2$$

one has the more general

[2]We postulate that we arrive at an equally justified description of the process if we refer it to an appropriately moving coordinate system; in that way we abide by the basic idea of the theory of relativity.

$$ds^2 = \sum_{ik} g_{ik} dx_i dx_k$$

as the absolute invariant (scalar), then one sees at once how one attains a generalization of the theory of relativity that encompasses gravitation on the basis of the equivalence hypothesis. While in the original theory of relativity the independence of the physical equations from the special choice of the reference system is based on the postulation of the fundamental invariant $ds^2 = \sum_i dx_i^2$, we are concerned with constructing a theory in which the most general line element of the form

$$ds^2 = \sum_{ik} g_{ik} dx_i dx_k$$

plays the role of the fundamental invariant. The concepts of vector analysis needed for that purpose are provided by the method of the absolute differential calculus,
[7] which will be explained in the lecture by Grossmann which is to come next.

It follows from the idea outlined above that the ten quantities g_{ik} characterize the gravitational field; they replace the scalar gravitational potential φ of Newtonian gravitation theory, and form the second-rank fundamental covariant tensor of the gravitational field. The fundamental physical significance of these quantities g_{ik} consists, i.a., in the fact that they determine the behavior of measuring rods and clocks.

The method of the absolute differential calculus allows us to generalize the systems of equations of any physical process, as they occur in the original theory of relativity, in such a way that they fit into the scheme of the new theory. The components g_{ik} of the gravitational field always appear in these equations. The physical meaning of this is that the equations provide information about the influence of the gravitational field on processes in the region under study. The previously indicated law of motion of the material point may serve as the simplest example of this kind. Otherwise, we shall confine ourselves to the formulation of the most general law known to physics, namely, the law that corresponds to the momentum and energy conservation law in the original theory of relativity. As is well known, one has there a symmetric tensor $T_{\mu\nu}$, the components of which, the stress components, yield the components of the momentum, and the components of energy flux density and energy density. These quantities can be specified for phenomena in any domain. The laws of momentum and energy conservation are contained in the equations

$$(1) \qquad \sum_\nu \frac{\partial T_{\sigma\nu}}{\partial x_\nu} = 0, \qquad (\nu,\sigma = 1,2,3,4)$$

since by integrating with respect to the spatial coordinates over the whole system, one can obtain from these equations the conservation equations

$$(1a) \qquad \frac{d}{dt}\left(\int T_{\sigma 4} d\tau\right) = 0,$$

where $d\tau$ denotes the three-dimensional volume element.

In the general theory, the following equations correspond to equations (1):

$$(2) \qquad \sum_\nu \frac{\partial \mathfrak{T}_{\sigma\nu}}{\partial x_\nu} = \frac{1}{2} \sum_{\mu\nu\tau} \frac{\partial g_{\mu\nu}}{\partial x_\nu} \gamma_{\mu\tau} \mathfrak{T}_{\sigma\nu} \qquad (\sigma = 1,2,3,4)$$ [8]

Here

$$\mathfrak{T}_{\sigma\nu} = \sqrt{-g} \cdot \sum_\mu g_{\sigma\mu} \theta_{\mu\nu},$$

where g is the determinant $|g_{ik}|$, and $\gamma_{\mu\tau}$ is the subdeterminant adjoint to $g_{\mu\tau}$ divided by this determinant; $\Theta_{\mu\nu}$ is the symmetrical second-rank contravariant tensor that characterizes the behavior of energy in the domain of phenomena under consideration. The quantities $\mathfrak{T}_{\sigma\nu}$ have the same physical meaning here as the quantity $T_{\sigma\nu}$ in the original theory of relativity; the stress-energy components of the gravitational field are not contained in them.

The right-hand side of equations (2) vanishes if the quantities $g_{\mu\nu}$ are constant, i.e., if no gravitational field is present. In that case, equation (2) reduces to equation (1) and can therefore be brought into the form (1a); in other words: the material process satisfies the conservation laws all by itself. If, on the contrary, the $g_{\mu\nu}$ are variable, i.e., if a gravitational field is present, then the right-hand side of equations (2) expresses the energetic influence of the gravitational field on the material process. It is clear that no conservation laws can be deduced from equation (2) in that case, because the stress-energy components of the material process cannot satisfy any conservation laws all by themselves, without the components of the gravitational field.

The method sketched up to this point shows how the equation systems of physics can be obtained when the influence of a given gravitational field on the processes is taken into account. But this does not solve the main problem of the theory of gravitation, since the latter consists in determining the quantities g_{ik} when the field-generating material processes (including the electrical ones) are to be considered as given. In other words, the generalization of Poisson's equation

$$(3) \qquad \Delta\varphi = 4\pi k \rho$$

is sought.

On the one hand, the proportionality of energy and inertial mass that is obtained from the ordinary theory of relativity, and, on the other hand, the empirical proportionality of inertial and gravitational mass lead necessarily to the view that the same quantities that determine the energetic behavior of a system must also determine the gravitational effects of the system. From this we conclude that tensor $\mathfrak{T}_{\mu\nu}$ must appear in equations of gravitation we are seeking, in lieu of the density ρ of equation

(3). We are therefore looking for equations that express the equality of two tensors, one of which is the given tensor $\mathfrak{T}_{\mu\nu}$, while the other comes from the fundamental tensor $g_{\mu\nu}$ through differential operations.

[9] It has now turned out that the conservation laws of momentum and energy make possible the derivation of these equations. It has already been emphasized above that the material process alone cannot satisfy the conservation laws; but we must demand that the conservation laws be satisfied for the material process and the gravitational field *together*. According to the arguments presented above, this means that there must exist four equations of the form

$$\sum_{\nu} \frac{\partial}{\partial x_\nu}(\mathfrak{T}_{\sigma\nu} + \mathfrak{t}_{\sigma\nu}) = 0. \qquad (\sigma = 1,2,3,4) \tag{4}$$

Here the $\mathfrak{t}_{\sigma\nu}$ characterize the stress-energy components of the gravitational field in a manner analogous to the way in which the quantities $\mathfrak{T}_{\sigma\nu}$ characterize those of the material process. In particular, the quantities $\mathfrak{T}_{\sigma\nu}$ and $\mathfrak{t}_{\sigma\nu}$ must have the same invariant-theoretical character. It turned out to be possible to show by means of a general argument that the equations that completely determine the gravitational field
[10] cannot be covariant with respect to arbitrary substitutions. This fundamental discovery is especially noteworthy because all other physical equations, such as, e.g., equations (2), possess general covariance. In accordance with this general result, the postulated equations (4) are also covariant only with respect to *linear* substitutions, but are not
[11] so with respect to arbitrary substitutions. Hence, we will have to demand covariance only with respect to linear transformations from the gravitation equations that we are seeking. It has turned out that one is led to completely determined equations if one adds to these considerations the demand that when these equations are applied to the relevant special case and an approximate solution is sought, they must yield Poisson's
[12] equation (3). Using the way indicated, one obtains the following equations:

$$\sum_{\alpha\beta\mu} \frac{\partial}{\partial x_\alpha}\left(\sqrt{-g}\,\gamma_{\alpha\beta}\, g_{\sigma\mu} \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\right) = \kappa(\mathfrak{T}_{\sigma\nu} + \mathfrak{t}_{\sigma\nu}); \qquad (\sigma, \nu = 1,2,3,4) \tag{5}$$

Here

$$-2\kappa \cdot \mathfrak{t}_{\sigma\nu} = \sqrt{-g}\left(\sum_{\beta\tau\rho} \gamma_{\beta\nu} \frac{\partial g_{\tau\rho}}{\partial x_\sigma} \frac{\partial \gamma_{\tau\rho}}{\partial x_\beta} - \frac{1}{2} \sum_{\alpha\beta\tau\rho} \delta_{\sigma\nu}\, \gamma_{\alpha\beta} \frac{\partial g_{\tau\rho}}{\partial x_\alpha} \frac{\partial \gamma_{\tau\rho}}{\partial x_\beta}\right); \tag{6}$$

κ is a universal constant that corresponds to the gravitational constants; $\delta_{\sigma\nu}$ is 1 or 0, depending on whether σ and ν are different or equal.

One can see from the system of equations (5), which corresponds to equation (3), that along with the stress-energy components $\mathfrak{T}_{\sigma\nu}$ of the material process, those of the gravitational field (namely, $\mathfrak{t}_{\sigma\nu}$) appear as an equivalent field-inducing cause, a circumstance that obviously must be demanded; for the gravitational effect of a system may not depend on the *physical nature* of the system's field-producing energy.

Since only linear substitutions are admissible, certain one-, two-, and three-dimensional manifolds are privileged, which may be designated as straight lines, planes, and linear spaces. [13]

The theory sketched here overcomes an epistemological defect that attaches not only to the original theory of relativity, but also to Galilean mechanics, and that was especially stressed by E. Mach. It is obvious that one cannot ascribe an absolute [14]
meaning to the concept of acceleration of a material point, no more so than one can ascribe it to the concept of velocity. Acceleration can only be defined as relative acceleration of a point with respect to other bodies. This circumstance makes it seem senseless to simply ascribe to a body a resistance to an acceleration (inertial resistance of the body in the sense of classical mechanics); instead, it will have to be demanded that the occurrence of an inertial resistance be linked to the relative acceleration of the body under consideration with respect to other bodies. It must be demanded that the inertial resistance of a body could be increased by having unaccelerated inertial masses arranged in its vicinity; and this increase of the inertial resistance must disappear again if these masses accelerate along with the body. It turns out that this behavior of inertial resistance, which we may call *relativity of inertia*, actually follows from equations (5). This circumstance constitutes one of the [15]
strongest pillars of the theory sketched.