Lecture 5

Geometrization of Newtonian Gravity

I. Geometrization of free body motion in an accelerated frame

II. Geometrization of gravity
1. The Equivalence Principle
2. Gravity geometrized via mathematized inertial motion

III. Looking ahead

For grasping what is equivalent in the "Equivalence Principle," read
(i) the caption to the Eötvös experiment on page 26 of A JOURNEY INTO GRAVITY AND SPACETIME by John A. Wheeler (ISBN 0-7167-5016-3 and/or
(ii) Box 38.2 in MTW

(iii) Einstein's 1913 1st attempt at a theory of gravition in "Physical Foundations of a Theory of Gravition" [available on the internet].

5.1

I. Geometrization of Free Body Motion in an Accelerated Frame

As before, consider the motion of a free particle in an inertial reference frame. Because of Newton's 1st Law its world line is one which is straight relative to that frame's rectilinear x-t coordinate system.

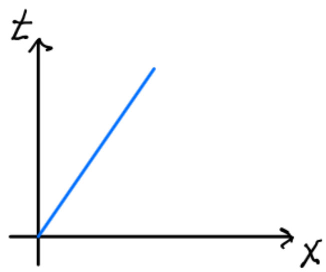


Figure 5.1 Free particle world line observed in an inertial frame of reference, relative to which

$$\frac{d^2x}{dt^2}=0$$

and

$$x(t)=v_0t. \qquad (5.1)$$

Newton's 1st Law does not apply relative to a uniformly accelerated reference frame. As

(5.2)

depicted in Figure 5.2, the free particle's 1-d spatial trajectory is observed to make a sharp U-turn relative to such a frame.

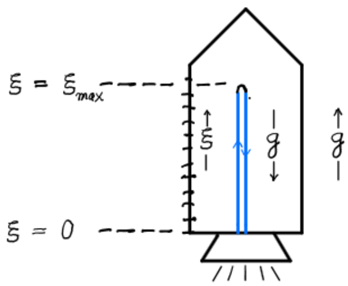


Figure 5.2 Motion of a freely moving particle as observed in a frame with acceleration of magnitude g upward.

Wordlines which are observed to be straight relative to an inertial frame of reference are observed to be curved relative to an accelerated frame such as the one depicted in Figure 5.2.

The equation motion and its solution* are

$$\frac{d^2x}{dt^2}=0 \qquad (5.2a)$$

and

5.3

*\footnote{ $C(t)=\begin{pmatrix}x\\t\end{pmatrix}=\begin{pmatrix}v_0 t\\t\end{pmatrix}$ }

$$x(t)=v_0 t, \qquad (5.2b)$$

relative to the inertial frame; by contrast, relative to the accelerated frame they are

$$\frac{d^2\xi}{d\tau^2}=-g \qquad (5.3a)$$

and

$$\xi(\tau)=v_0\tau-\tfrac{1}{2}g\tau^2 \qquad (5.3b)$$

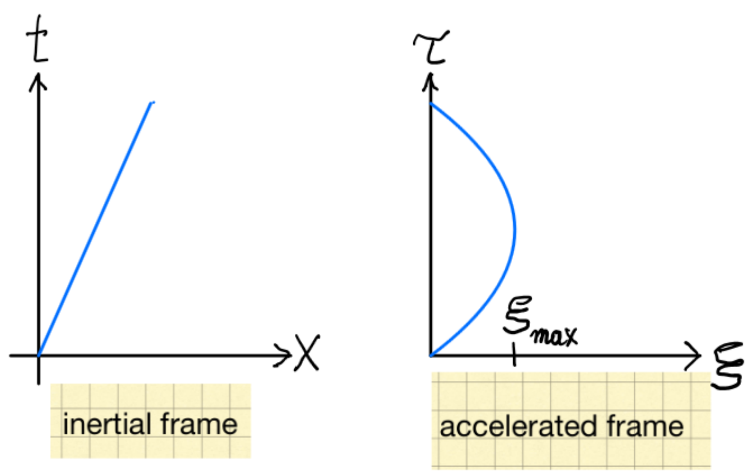
Each of these two frames is a platform for measurements performed on the worldline of a free particle. These frames are mathematized by the (x,t) and the (ξ,τ) coordinate charts (a.k.a. coordinates systems) respectively. The coordinates are the standards of measurement for each frame. The transition map* (a.k.a. coordinate transformations) between the two charts is

*\footnote{ $\varphi_{in}\circ\varphi_{ac}^{-1}$ }

$$x=\xi+\tfrac{1}{2}g\tau^2 \qquad (5.4)$$
$$t=\tau$$

(5,4)

This transformation relates the free particle world line represention relative to the inertial frame to that relative to the accelerated frame. Figure 5.3 depicts these two representions of one and the same motion of a particle.



\begin{figure}

Figure 5.3 The transformation Eq. (5.4),

$$\begin{pmatrix} \xi \\ \tau \end{pmatrix} \rightsquigarrow \begin{pmatrix} x = \xi + \frac{1}{2} g \tau^2 \\ t = \tau \end{pmatrix}$$

(5.5)

maps the domain of the accelerated frame with its curvilinear (ξ, τ) coordinatization into the inertial frame with its rectilinear (x, t) coordinatization:

$$(\xi, \tau) \rightsquigarrow (x, t) = \left(v_0 \tau + \tfrac{1}{2} g \tau^2 ;\ \tau\right) \tag{5.5}$$

Furthermore, it maps the free-particle equation of motion

$$\frac{d^2 x}{dt^2} = 0$$

and its straight world line solution*

*\footnote{ To be math'lly precise $C_{in}(t) = \begin{pmatrix} C^0(t) \\ C^1(t) \end{pmatrix} = \begin{pmatrix} t \\ v_0 t \end{pmatrix}$ }

$$x(t) = v_0 t$$

in the inertial frame onto the particle equation of motion

$$\frac{d^2 \xi}{d\tau^2} = -g$$

and its solution*

*\footnote{ To be mathematically precise $\left(\varphi_{acc} \circ \varphi_{in}^{-1} \circ C_{in}\right)(\tau) = C_{acc}(\tau) = \begin{pmatrix} C^0_{acc}(\tau) \\ C^1_{acc}(\tau) \end{pmatrix} = \begin{pmatrix} \tau \\ v_0 \tau - \frac{1}{2} g \tau^2 \end{pmatrix}$ }

$$\xi(\tau) = v_0 \tau - \tfrac{1}{2} g \tau^2$$

in the accelerated frame.

In summary,
the transformation $\boxed{\text{Eq.(5.5)}}$ maps the two representations of the particle world line, (5.1) and (5.2), onto one another:

\end{figure}

Thus for a freely moving body of mass m, its acceleration and hence the force impelling it as observed relative to an accelerating frame is

$$m\frac{d^2\xi}{d\tau^2} = -mg \qquad (5.4)$$

This force, like the Coriolis and centrifugal force, is an <u>inertial pseudo</u> force. "inertial" because it is proportional to the mass; "pseudo" because it comes from the non-inertial (here, accelerated) frame of reference.

(Lecture 4) 5.7

Equation (5.4), just like Eq. (4.3) lendsitself to being geometrized by comparing it with the equation for a geodesic in spacetime.

This is achieved by comparing

(a) the free-body equation of motion measured in an accelerated frame and mathematized by the second derivative of Eq. (5.2),

$$\frac{d^2\xi}{d\tau^2} = -g \qquad (5.5)$$

with

(b) the $\mu=1$ component of the geodesic eq'n

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

by setting $x^1=\xi$. In the asymptotic non-relativistic (low velocities: $\left(\frac{dx^1}{d\tau}\right)^2 \ll 1$)) limit

$\frac{dx^0}{d\tau}=1$ and hence $\tau = c\, t_{conv}$.

Consequently, (5.8)

$$\frac{d^2\xi}{d\tau^2} = -\Gamma^1_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

becomes

$$\frac{d^2\xi}{d\tau^2} = -\Gamma^1_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} + \text{neglegible terms.} \quad (5.6)$$

Comparison with Eq. (5.5) leads to.

$$\boxed{\Gamma^1_{00} = g = \frac{g_{conv}}{c^2}} \quad (5.7)$$

and

$$\Gamma^0_{00} = \Gamma^0_{01} = 0$$

because $\frac{d}{d\tau}\left(\frac{dx^0}{d\tau}\right) \approx 0.$

II. Geometrization of Gravity

1. Q: Must one formulate gravity in geometrical terms?

A: The following is an observed fact: the inertial pseudo force on a particle moving freely

in a uniformly accelerated frame is (5,9)
indistinguishable from the force
on a particle in a uniform gravitational field.
This fact is mathematized by the statement that
\footnote{Recall that the acceleration in conventional unit, $g_{conv}\left[\frac{length}{(time)^2}\right]$ is related g in standard units by

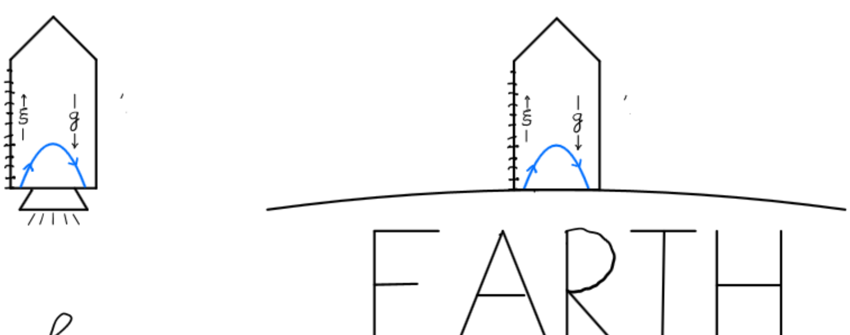
$$g = \frac{g_{conv}}{c^2}\left[\frac{1}{length}\right].$$

Furthermore, the dimensionless grav'l pot'l ϕ is related to the conventional Newtonian grav'l pot'l ϕ_{conv} by $\phi = \frac{\phi_{grav}}{c^2}$.}

$$\boxed{m_{inert}\, g_{conv} = m_{grav'l}\left(-\vec{\nabla}\phi_{grav'l}\right)^{i=1}} \qquad (5,8)$$

or

$$\boxed{m_{inertial}\, g = m_{grav'l}\left(-\vec{\nabla}\phi\right)^{i=1}}$$



where

$m_{inertial} = m_{gravitational}$

Taken from *A Journey Into Gravity And Spacetime* by John A. Wheeler

EÖTVÖS'S EXPERIMENT

To measure the travel time of a falling body more precisely demands more time. So realized Baron von Eötvös, who therefore devised an experiment of a new kind that offered unlimited time for measurement. It focused on the central issue: Is there for any substance any such distinction, as old writers assumed, between its "gravitational mass," on which the center of the Earth is conceived to pull, and its "inertial mass"? The "inertial mass"—today simply "mass"—resists being set in motion or, if already endowed with a velocity, resists any change in the magnitude or the direction of that velocity.

Imagine an object endowed with inertial mass but no gravitational mass, hanging at the end of a fiber and carried around and around in a circle by the spin of the Earth. With no gravity to hold it down, it will pull the fiber straight out from the axis (red line in the diagram). On the other hand, an object that has no inertial mass at all will not be thrown outward from the axis of spin of the Earth. Instead—if it has any gravitational mass—it will tug the supporting fiber to a position where it points straight down to the center of the Earth (blue line). An object deprived neither of gravitational mass nor of inertial mass will pull the supporting fiber neither straight out nor straight down, but instead to an angle of uprise (green line). Does the ratio of inertial to gravitational mass differ from one substance to another? Then the angle of uprise will differ, too. And for measuring it there's all the time in the world!

This beautiful idea Eötvös put into action. He and his colleagues, in experiments extending over some thirty years, on a variety of substances, were able to establish that there is an angle of uprise: that angle is greatest at latitudes 45°N and 45°S, and amounts there to a tenth of a degree. They found to an accuracy of 5 parts in 10^9 that the angle of uprise was identical for every substance tested.

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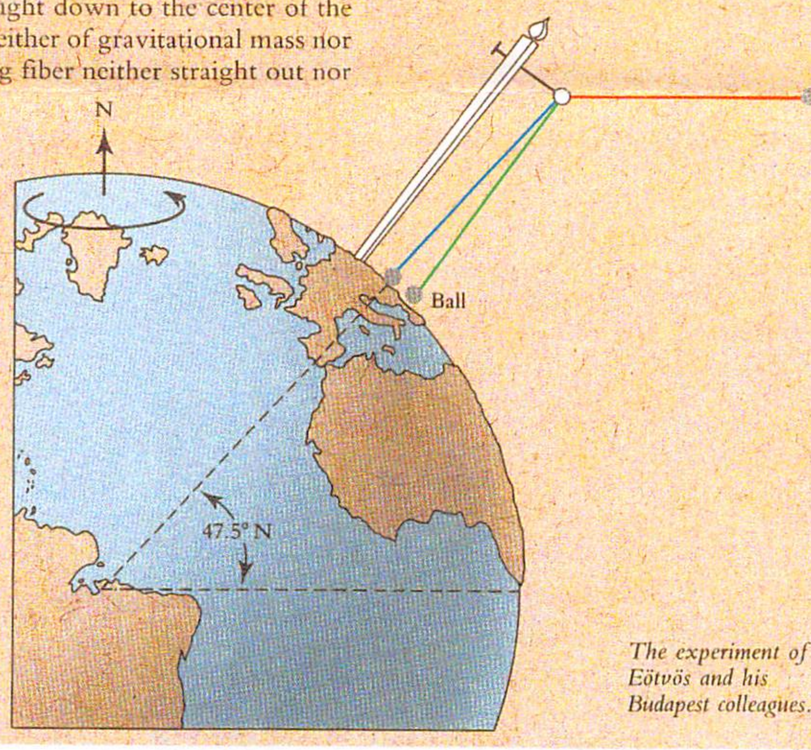
Born July 27, 1848, Budapest. Died April 8, 1919, Budapest.



The experiment of Eötvös and his Budapest colleagues.

for all kinds of material particles. (5.10)

(Eötvös experiment)
accuracy as of 2017: 1 part in 10^{15}

2. The line of reasoning leading to the conclusion

"Geometrized free-body motion + Equivalence principle = Geometrized gravity"

is a four-step process

(i) The gravitational force field is conservative implies

$$\overrightarrow{(\text{grav'l force})} = m_{grav}\,(-)\,\vec{\nabla}\phi_{grav} \,,$$

where ϕ_{grav} is the Newtonian gravitational potential.

\footnote{Example: For a spherical body of mass M

$$\phi_{grav}(x,y,z) = -\frac{GM}{r} = -GM\frac{1}{\sqrt{x^2+y^2+z^2}}$$ }

(ii) Apply the equivalence principle to the one-dimensional geometrized motion, Eq. (5.5) - (5.8) on pages 5.3 - 5.5: (5.11)

$$m_{in}\left(\frac{d^2\xi}{dt^2_{conv}}\frac{1}{c^2} \overset{(5.7)}{\doteq} -\frac{g_{conv}}{c^2} \overset{(5.6)}{\doteq} -\Gamma^1_{00}\right)$$

$$\overset{(5.8)}{\doteq} m_{grav}\left(-\frac{1}{c^2}(-)\left(\vec{\nabla}\phi_{grav}\right)^i\right) \qquad (5.8)$$

★ \footnote{Recall that

see below

$$\Gamma^i_{00} = \frac{1}{2}\sum_{\nu=0}^{3} g^{i\nu}\left(g_{\nu 0,0} + g_{\nu 0,0} - g_{00,\nu}\right)$$

a) Focus on time-independent acceleration. Thus

$$g_{\nu 0,0} = g_{00,0} = 0$$

b) Use rectilinear coordinates. Thus

$$g^{ij} = \delta^{ij} \qquad \begin{matrix} i \\ j \end{matrix}\Big\} = 1,2,3$$

c) Consequently,

$$\Gamma^i_{00} = -\frac{1}{2} g_{00,i} = -\frac{1}{2}\left(\vec{\nabla} g_{00}\right)^i$$

(iii) Introducing* Γ^i_{00} into Eq. (5.8) (5.12)
yields
$$\frac{1}{2}\left(\vec{\nabla} g_{00}\right)^i = \frac{m_{grav}}{m_{inert}} \frac{-1}{c^2}\left(\vec{\nabla}\phi_{grav}\right)^i$$

Using the Eötvös fact that
$$\frac{m_{grav}}{m_{inert}} = 1$$
results in
$$\vec{\nabla} g_{00} = -\frac{2}{c^2}\left(\vec{\nabla}\phi_{grav}\right)$$
and
$$g_{00} = const - 2\,\frac{\phi_{grav}}{c^2}$$

(iv) Impose the observed boundary condition that in the absence of gravitation (i.e. $\phi_{grav} = 0$) the spacetime metric is flat, which means there exist a global coordinat system such that
$$-d\tau^2 = g_{00}\,dt^2 + \sum_i \sum_j dx^i dx^j$$
$$= -dt^2 + dx^2 + dy^2 + dz^2,$$

Thus, $g_{oo} = -1$ whenever $\phi_{grav} = 0$.

It follows that

$$\boxed{g_{oo} = -1 - 2\frac{\phi_{grav.}}{c^2} = -1 - \frac{2}{c^2}\left(\begin{array}{c}\text{Newtonian}\\ \text{gravitational}\\ \text{potential}\end{array}\right)}$$

CONCLUSIONS

1. Gravitation is to be mathematized in terms of geometrical concepts. The gravitational potential is the Newtonian limit of a geometrical formulation in terms of the metric tensor

2. The five step line of reasoning implies that fundamentally

$$\left(\begin{array}{l}\text{set of generalized}\\ \text{gravitational}\\ \text{potentials}\end{array}\right) = \left(\begin{array}{l}\text{set of components}\\ \text{of a metric}\\ \text{tensor field}\end{array}\right)$$

or more briefly

5.14

$\boxed{\text{"gravitation = geometry"}}$ (properly understood)

3.

$$\boxed{\begin{array}{c}\text{Newton's } 1^{st} \text{ Law}\\ \text{geometrized relative}\\ \text{to rotating \& acc'd frames}\end{array}} \rightarrow \begin{Bmatrix}\text{inertial}\\ \text{forces}\end{Bmatrix} = \left\{\Gamma^{i}_{\alpha\beta}\right\}$$

$\downarrow$ Equivalence Principle $\rightarrow \Gamma^{i}_{00} = \frac{1}{c^2}\left(\vec{\nabla}\phi_{grav}\right)^{i}$

$$\boxed{\text{Gravity} \neq 0 \implies \begin{array}{l}\text{Metric tensor}\\ \text{is not flat}\end{array}}$$

4. What is the geometrized generalization of Newton's gravitational field equations

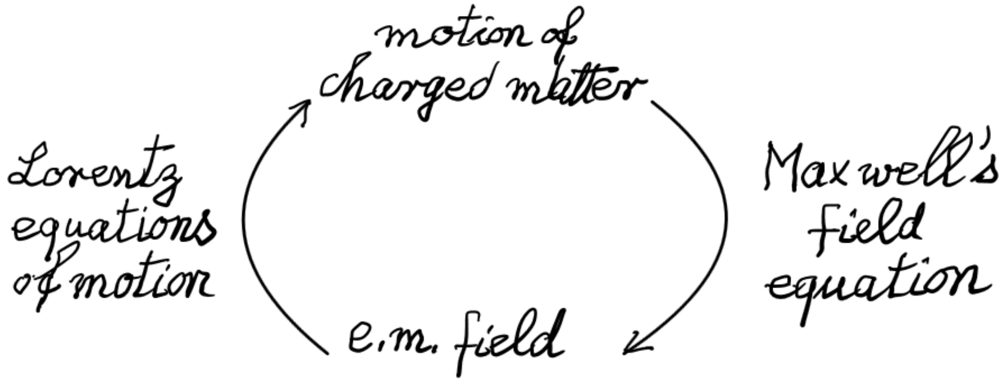
$$\nabla^2\phi = 4\pi\rho \,?$$

Here ρ is the mass density

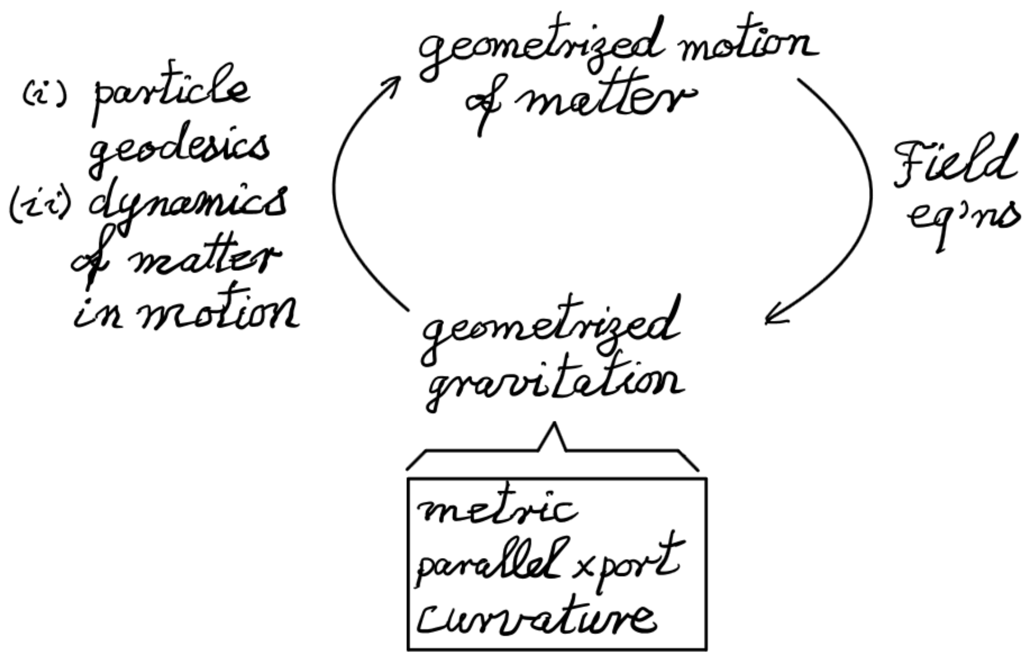
5.15

III. Looking ahead we have the

1. Mathematical Structure of Eletromagnetism



2. Mathematical Structure of Gravitation



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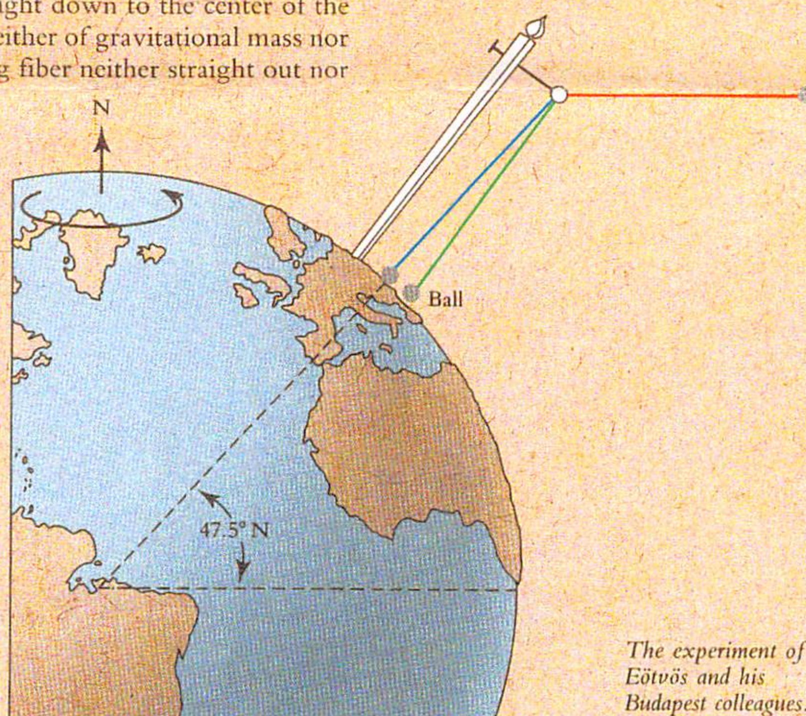

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