LECTURE 6

MATTER and MOTION

I. Important thinkers

II. Momentum
1. Newtonian
2. Relativistic

Read Chapter 2 in the 1st Edition of
SPACETIME PHYSICS
by Taylor & Wheeler

OR Chapter 7 in the 2nd Edition of
SPACETIME PHYSICS
by Taylor and Wheeler

Chapter 6 in A JOURNEY INTO
GRAVITY AND SPACETIME
by J.A. Wheeler

(6.1)

In order to mathematize gravitation one must mathematize the motion of bodies, more generally of matter. This is because gravitation leaves its perceptible imprints in the form of the motion of matter. This is to be contrasted with electromagnetism whose imprints of (charged) matter motion is entirely different.

Subsequently we shall mathematize (by means of the Einstein field equations) gravitation as a response to matter in motion.

I. Change, Matter, Motion, and its Causes

The first thinkers to identify the concepts of matter and of motion where the Greek philosophers, even before Socrates (Thales, Heraclitus, Parmenides,

6.2

Zeno, Democritus, ...) and then most importantly Aristotle.* He reconciled Parmenides's and Heraclitus's seemingly irreconcilable conclusions about the phenomenon of change. He* pointed out that change presupposes the law of identity, namely: a thing is what it is; its characteristics constitute its identity.

Using his law of identity he* showed there are four causal factors involved in change: 1. the material cause, 2. the formal cause, 3. the efficient cause, 4. the final cause. Ayn Rand (1905-1982) identified the basis of causality in the law of identity by observing that the law of causality is the law of identity applied to action.

*\footnote{ An exposition and explanation of Aristotle's fundamental work on change and causality — which I found to be second to none in clarity combined with pithiness can be found by "googling" "Peikoff Aristotle metaphysics." }

The concepts of change and motion were put into mathematical form by Galileo, Kepler, and most definitively by Newton with his laws of dynamical motion. They are the platform from which all future developments where launched. His familiar fundamental dynamical laws are three in number:

1. Every object in a state of uniform motion tends to remain in that unless a force is applied to it.
2. $\overrightarrow{\text{Force}} = \frac{d}{dt}(\overrightarrow{\text{momentum}})$
3. For every action there is an opposite and equal reaction.

These three laws are integrated into an organic whole by the law of the conservation of momentum in a collision process.

Using Special Relativity, Einstein not only generalized Newton's dynamical laws of motion to relativistic velocities, but he also geometrized them.

(6.3)

He did this by introducing the integrating concept of the <u>energy-momentum</u> (better known as "<u>momenergy</u>") <u>four-vector</u> into the dynamics of relativistically moving matter.

II. Momentum

1. Arrive at the definition of momentum by focusing on low velocity collisions by means of the following Newtonian line of reasoning. Thus ask the question:

What quantity common to all collision processes remains the same before and after any such process?

Answer:

a) Consider the following spacetime

process involving the collion (6.4)
of two particles.

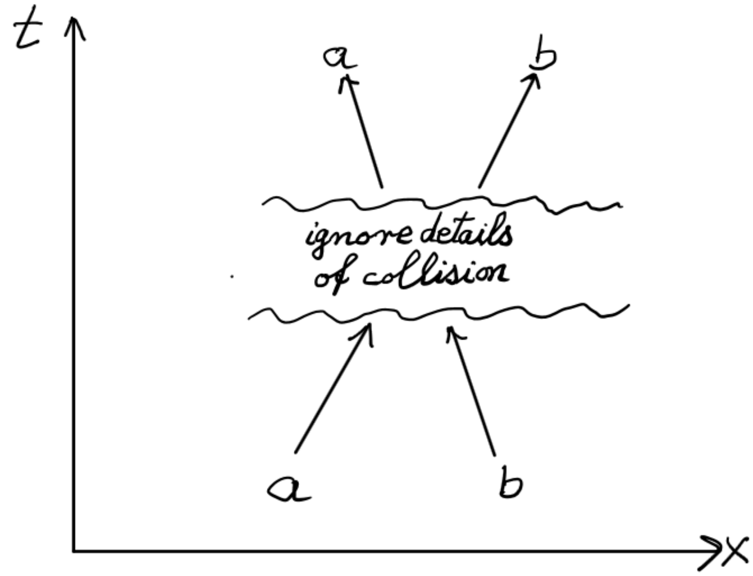

Figure 4.1 Collision between particles a and b.

b) Consider the quantity

$$(\text{mass}) \times (\text{velocity}) \equiv \text{Momentum}$$

c) Focus on Newton's 2^{nd} Law

$$\frac{d}{dt}(\text{Momentum of } a) = F_{b\,on\,a}$$

d) Take notice of Newton's 3rd Law (6,5)

$$F_{b\,on\,a} = -F_{a\,on\,b}$$

e) Apply c) & d) to all particles and conclude after integration that

$$\sum_{\substack{\text{particles}\\ a,b}} (\text{Momentum before}) = \sum_{\substack{\text{particles}\\ a,b}} (\text{Momentum after})$$

2. Generalize the concept of momentum to relativistic:

A. Consider (mass)(velocity) and then label

$$m\,\frac{d\vec{v}}{dt} = \overrightarrow{\text{momentum}}$$

as "momentum". However, this does not lead to any conceptual economy. This is because such an extension of the Newtonian definition does not lead to the expression for any conserved quantity before and after when examined in different inertial frames.

(6,6)

Hence focus attention on those quantities that do exhibit conservation.

B. Definition of relativistic momentum as identified by Tolman & Ehrenfest.

(FYI: A definition is the condensation of a vast body of observations - and stands or falls with the truth or falsehood of the observations.*)

*\footnote{See Chapter 5 ("Definitions") in 2nd Edition of "INTRODUCTION TO OBJECTIVIST EPISTEMOLOGY" by Ayn Rand}

Theorem (Definition of momentum)

GIVEN: a) Principle of Relativity
b) Isotropy of space
c) Symmetry

d) There exists a unique momentum vector associated with a particle having a given velocity (6,7)

e) The momentum is conserved during a collision

f) The correspondence at low velocities with Newton's definition must not be violated

CONCLUSION:

$$\text{momentum} = m \frac{\vec{\beta}}{\sqrt{1-\beta^2}}; \quad \vec{\beta} = \text{velocity}$$

Comment about the proof:

1. Look at every symmetric collision from the point of view of two reference frames
2. Look for a quantity conserved in both frames to find the momentum-velocity relationship.

Proof in six steps:

1. Q: What is the relation between
   $\vec{p}$ = momentum, and
   $\vec{v}$ = velocity of the particle?

   A: $$\boxed{\vec{p} = f(v)\,\vec{v}}$$

   Why? (i) isotropy of space
   (ii) $\vec{p}$ is unique
   (iii) $f(v) = f\left(\sqrt{v_x^2 + v_y^2 + v_z^2}\right)$

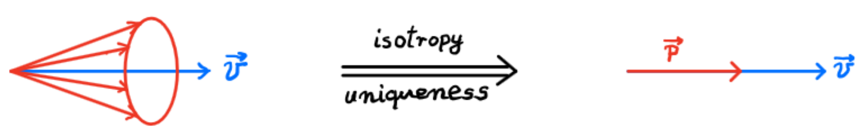

Figure 6.1 Momentum and velocity are necessarily collinear.

2. Principle of Relativity implies $f(v)$ is the <u>same</u> function in <u>all</u> inertial reference frames

3. To determine f, consider the elastic

collision between two identical particles viewed in to inertial frames. (6.9)

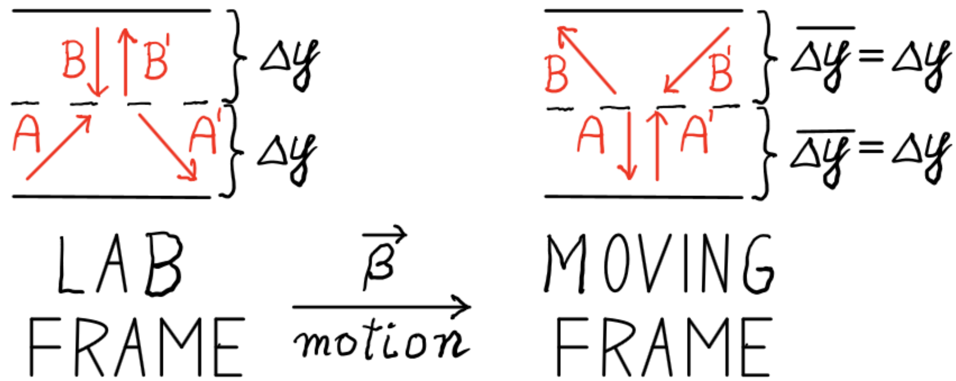

Figure 6.2 Collision observed in two different inertial frames.

For this type of a collision scenario one has

$$\left.\begin{array}{l}\bullet \text{ particles are identical} \\ \bullet \text{ Principle of Relativity} \\ \bullet \text{ isotropy of space}\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\text{the two} \\ \text{pictures are} \\ \text{symetrical}\end{array}\right.$$

- symmetry $\Longrightarrow$ pictures are <u>congruent</u>!

Consequently,

$\vec{A}'$ in LAB $= -\vec{B}$ in MOVING FRAME
$\vec{B}'$ in LAB $= -\vec{A}$ in MOVING FRAME

6.10

4. (i) In the MOVING FRAME the components of A are

A's velocity $= \dfrac{\overline{\Delta y}}{\overline{\Delta t}}$, where $\overline{\Delta t}$ = time for A to move from bottom to the point of collision

A's four-vector $= (\overline{\Delta t}, 0, \overline{\Delta y}, 0)$.

(ii) In the LAB FRAME the components of A are:

A's four-vector $= (\Delta t, \Delta x, \Delta y, 0)$.

$$= \left(\frac{\overline{\Delta t}}{\sqrt{1-\beta^2}}, \frac{\beta \overline{\Delta t}}{\sqrt{1-\beta^2}}, \overline{\Delta y}, 0\right)$$

A's spatial velocity $= \left(\dfrac{\Delta x}{\Delta t}, \dfrac{\Delta y}{\Delta t}, 0\right)$

$$= \left(\beta, \frac{\overline{\Delta y}}{\overline{\Delta t}}\sqrt{1-\beta^2}, 0\right)$$

$$\equiv \left(\beta, \mu\sqrt{1-\beta^2}, 0\right)$$

where $\mu = \dfrac{\overline{\Delta y}}{\overline{\Delta t}}$

5.) Conservation of momentum in LAB FRAME:

6.11

(i) Along the x-direction

$$\sum p_x)_{before} = \sum p_x)_{after}$$

Referring to Figure 6.2, find that this equality becomes

$$\underbrace{f\left(\sqrt{\beta^2+\mu^2(1-\beta^2)}\right)\times\mu\sqrt{1-\beta^2}}_{A}\ \underbrace{-f\left(\sqrt{0+\mu^2}\right)\mu}_{B} =$$

$$= \underbrace{f\left(\sqrt{\beta^2+\mu^2(1-\beta^2)}\right)\times(-)\mu\sqrt{1-\beta^2}}_{A'}\ \underbrace{+f\left(\sqrt{0+\mu^2}\right)\mu}_{B'}$$

The resulting functional identity is

$$f\left(\sqrt{\beta^2+\mu^2(1-\beta^2)}\right) = \frac{f(\mu)}{\sqrt{1-\beta^2}}$$

Take limit $\mu \to 0$ and obtain

$$\boxed{f(\beta) = \frac{f(0)}{\sqrt{1-\beta^2}}}$$

6.12

6.) The value of $f(0)$ is obtained from the Newtonian correspondence limit.

Asymptotically one has

$$f(v)\, v_y \longrightarrow m\, v_y$$

$$1 = \frac{p_y}{p_y} = \lim_{v \to 0} \frac{m v_y}{f(v) v_y} = \frac{m}{f(0)}$$

$$\therefore \quad f(\beta) = \frac{m}{\sqrt{1-\beta^2}}$$

Conclusion

$$\boxed{\overrightarrow{\text{momentum}} = \frac{m}{\sqrt{1-\beta^2}}\,\vec{\beta}}$$

Comment:

If $f(v)$ is independent of v then one implicitly assumes the Newtonian (non-relativistic) approximation.