LECTURE 7

I. Conservation of Energy

via Momentum Conservation + the Principle of Relativity

II. The Momenergy 4-vector

Read Sections 7.1-7.7

in SPACETIME PHYSICS (2^{nd} Edition)

by Taylor & Wheeler

Also very good is

Chapter 2 (MOMENTUM and ENERGY)

in the 1^{st} Edition of SPACETIME PHYSICS

I. Momentum, Energy, and their Conservation

(7.1)

In Relativity momentum and energy form an organic whole, namely momenergy. This concept is one in which momentum and energy are inextricably intertwined.

1. Momentum in Relativity

Collisions of particles is the observational basis for momentum and it conseration in Newtonian dynamics as well as its generalization to relativistic dynamics. In this context the 3-d relativistic momentum vector is

$$\overrightarrow{\text{momentum}} = \frac{m}{\sqrt{1-\beta^2}}\,\vec{\beta} \qquad (7.1)$$

2. Enery in Relativity

In Newtonian mechanics

7.2

momentum conservation and energy conservation are distinct physical principles. In relativistic mechanics, by contrast, momentum conservation <u>implies</u> energy conservation:

$$\text{tot. mom.} \equiv \sum_j \frac{m_j \vec{\beta}_j}{\sqrt{1-\beta_j^2}}\Bigg|_{\text{before}} = \sum_i \frac{m_i \vec{\beta}_i}{\sqrt{1-\beta_i^2}}\Bigg|_{\text{after}} \Longrightarrow \text{energy conservation}$$

The line of reasoning leading to this conclusion draws on two facts.

1. The unit tangent to the world line of a particle is a 4-vector with components

$$\{u^\alpha\} = \left\{\frac{dt}{d\tau}, \frac{d\vec{x}}{d\tau}\right\} \equiv \left\{\frac{dt}{d\tau}, \vec{u}\right\} \qquad (7.2)$$

whose Lorentz transformation property is known:

$$\boxed{\{u^\alpha\} \overset{\Lambda^{\bar{\alpha}}{}_{\alpha}}{\rightsquigarrow} \{\bar{u}^{\bar{\alpha}}\}.} \qquad (7.3)$$

2. The spatial 3-vector is proportional

to the relativistic 3-momentum (7.3)

$$\vec{u} \propto \frac{m\vec{\beta}}{\sqrt{1-\beta^2}}$$

Indeed, in light of

$$(d\tau)^2 = dt^2 - dx^2 - dy^2 - dz^2$$

$$1 = \left(\frac{dt}{d\tau}\right)^2 - \underbrace{\left(\frac{dx}{d\tau}\right)^2}_{\left(\frac{dx}{dt}\frac{dt}{d\tau}\right)^2 \equiv \beta_x^2\left(\frac{dt}{d\tau}\right)^2} - \left(\frac{dy}{d\tau}\right)^2 - \left(\frac{dz}{d\tau}\right)^2$$

Consequently,

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\beta^2}}, \quad \frac{d\vec{x}}{d\tau} = \frac{\vec{\beta}}{\sqrt{1-\beta^2}}$$

and Eq. (7.2) becomes

$$\{u^\alpha\} \equiv \{u^0, \vec{u}\} = \left\{\frac{1}{\sqrt{1-\beta^2}}, \frac{\vec{\beta}}{\sqrt{1-\beta^2}}\right\}. \qquad (7.4)$$

Compare the spatial components $\{u^x, u^y, u^z\} \equiv \vec{u}$ of Eq. (7.4) with (7.1) and find

$$\boxed{\vec{p} = m\vec{u}.} \qquad (7.5)$$

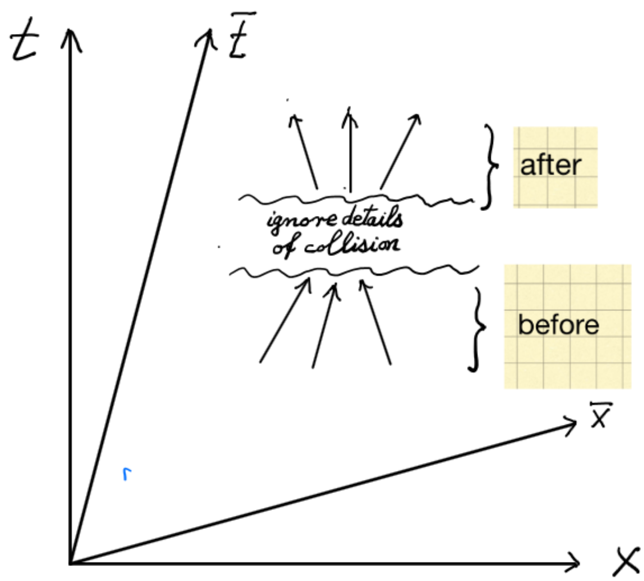
The relativistic 3-momentum is proportional, component by component, to $\vec{u}$, the spatial part of the 4-velocity with the mass m serving as the proportionality constant. 7.4

The spatial part

$$\vec{u} = \frac{\vec{\beta}}{\sqrt{1-\beta^2}} \qquad (7.6)$$

of the 4-velocity is the key that unlocks the door to the relativistic law of energy conservation.
Indeed, apply Eq. (7.5) to the conservation of momentum in two inertial frames in relative motion.

before



7.5

Figure 4.1 Collision between particles observed in two inertial reference frames. In the (x,t)-frame, during a collision total momentum is neither created nor destroyed; it is conserved. The total momentum is the same before and after the collision:

$$0 = \sum_i \vec{p}_{(i)}\Big|_{after} - \sum_j \vec{p}_{(j)}\Big|_{before}$$

Because this is a vectorial equality, it holds for each of its components separately.

The theorem on the next page makes this application explicit.

Theorem (Conservation of Energy) 7.6

GIVEN:

(a) The Principle of Relativity

(b) Conservation of spatial momentum in a collision of particles

CONCLUSION:

Total enery of particles is conserved

proof (in 4 steps):

1. Focus on a collision process in two inertial frames

Let $\{u^\mu\}$ refer to the 4-velocity of a particle.

2. Apply the transformation law to the momentum of each particle

$$p_x = m u_x = m(\bar{u}_x \, ch\theta + \bar{u}_t \, sh\theta)$$

$$= \bar{p}_x \, ch\theta + m \frac{1}{\sqrt{1-\beta^2}} sh\theta$$

7.7

3. Apply the momentum conservation principle to the particle momenta in each inertial reference frame

$$0 = \sum_i p_{x(i)}\Big|_{after} - \sum_j p_{x(j)}\Big|_{before}$$

$$= \underbrace{\left(\sum_i \bar{p}_{x(i)}\Big|_{after} - \sum_j \bar{p}_{x(j)}\Big|_{before}\right)}_{= 0} ch\theta$$

$$+ \left(\sum_i \frac{m_i}{\sqrt{1-\beta_i^2}}\Big|_{after} - \sum_j \frac{m_j}{\sqrt{1-\beta_j^2}}\Big|_{before}\right) sh\theta$$

Thus

$$\boxed{\sum_i \frac{m_i}{\sqrt{1-\beta_i^2}} \text{ is conserved.}}$$

4. Go to the Newtonian non-relativistic approximation $\left(\beta^2 \equiv \frac{v^2}{c^2} \ll 1\right)$ and find

$$\frac{mc^2}{\sqrt{1-\beta^2}} = mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots\right)$$

$$\simeq \underbrace{mc^2}_{\substack{\text{rest}\\\text{mass}\\\text{energy}}} + \underbrace{\tfrac{1}{2}mv^2 + \cdots}_{\substack{\text{kinetic}\\\text{energy}}} = \begin{pmatrix}\text{mass-energy}\\\text{of the particle}\end{pmatrix} \quad (7.8)$$

SUMMARY

Momentum conservation + P. of R.

$$\Rightarrow \boxed{\sum_i (\text{mass-energy})_i \equiv \text{Total mass-energy is conserved}}$$

II. The Energy-Momentum 4-vector

Keeping in mind that a definition is the condensation of a vast body observations – and stands or falls with the truth or falsehoods of these observations, arrive at the following

7.9

Definition

a) $\begin{pmatrix}\text{momenergy}\\ \text{for each}\\ \text{particle}\end{pmatrix} = \mathbb{p} : \{p^\alpha\} \equiv \left\{ \frac{m}{\sqrt{1-\beta^2}}, \frac{m\vec{\beta}}{\sqrt{1-\beta^2}} \right\}$ (7.7)

b) $\begin{pmatrix}\text{magnitude of}\\ \text{momenergy}\end{pmatrix}^2 = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2$

$= m^2 = (\text{rest mass})^2$

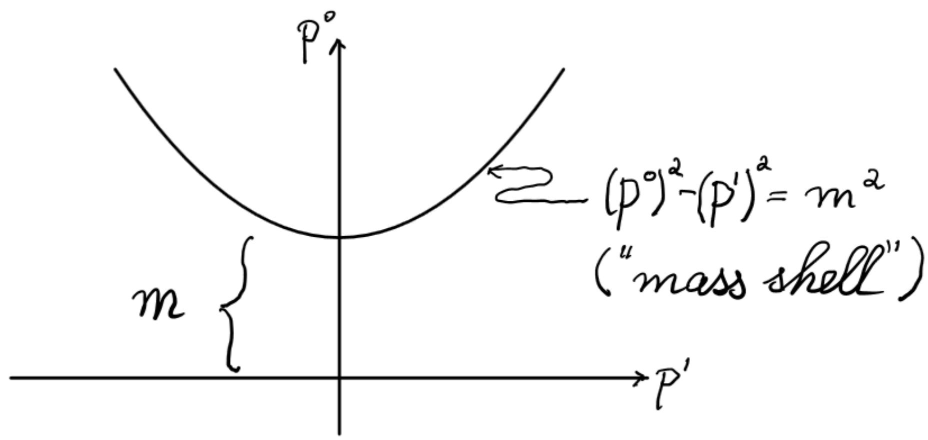


Figure 7.2 The momergy of a particle lies on its (rest) mass-shell.

c) Kinetic Energy

A particle whose observed momenergy components are given by Eq. (7.7)

7.10

on page 7.9 has kinetic energy

$$K.E. = \frac{m}{\sqrt{1-\beta^2}} - m = \frac{1}{2}m\beta^2 + \frac{3}{8}m\beta^4 + \frac{5}{16}m\beta^6 + \cdots$$

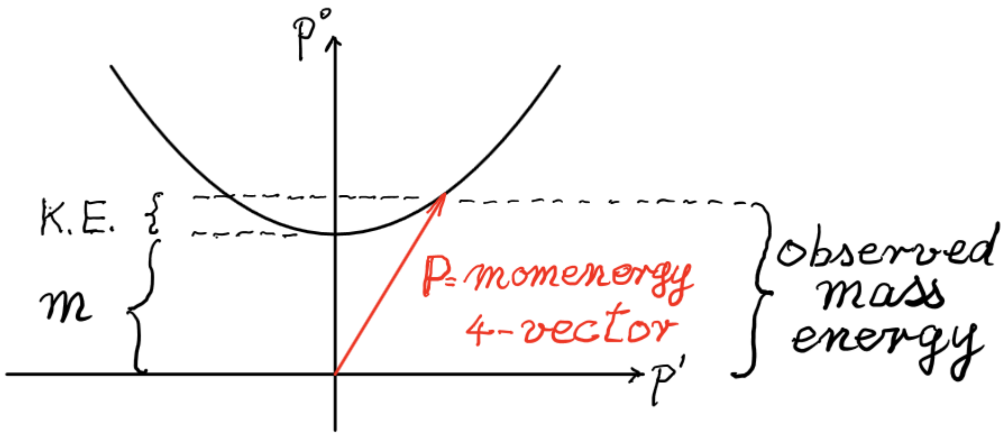


Figure 7.3 A particle with momergy 4-vector **P** (an element in the 4-d vector space of relativistic 4-momenta) and rest mass m has observed mass-energy

$$\frac{m}{\sqrt{1-\beta^2}} = m + K.E.$$

\footnote{ Every application of the law of momenergy conservation

in a collision process is a statement (7.11)
about a polygon built of 4-momenta
in the vector space of relativistic
4-momenta

$$\boxed{\sum_{i=1}^{N} \mathbb{P}_i\Big|_{before} = \mathbb{P}_{tot}\Big|_{before} = \mathbb{P}'_{tot}\Big|_{after} = \sum_{j=1}^{M} \mathbb{P}'_j\Big|_{after}}$$

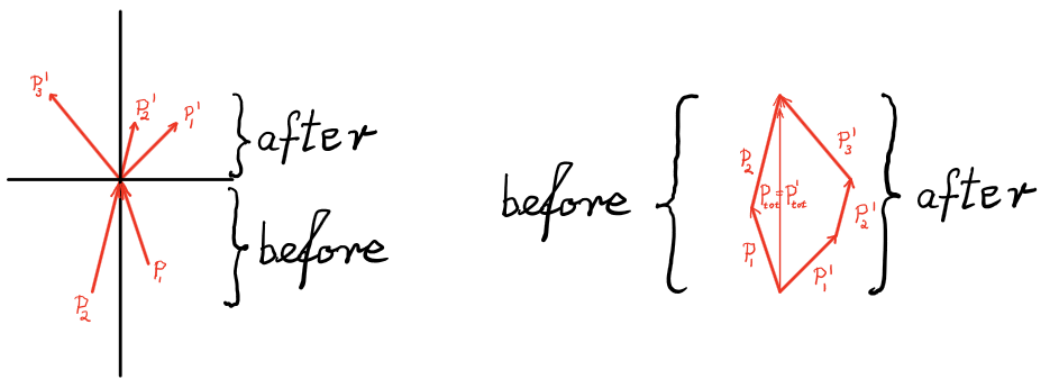


Figure 7.4 Collision event geometrized by a closed polygon in the vector space of 4-momenta

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