Lecture 8

Particle density-flux

A. COMOVING frame vs. LAB frame

B. 3-volumes in spacetime

In MTW know Box 4.4 (the math of $^*J$)

Box 5.2 (the physics of $^*J$)

Box 15.2 (the charge density-flux 3-form)

I. "Particle" as a Fundamental Concept in Physics (8.1)

In physics a particle is a contextually small entity of a specific nature and having specific attributes (mass, momenergy, spin, charge, color, etc.). It is the building block fundamental to understand the physical world.

Matter, in whatever state of motion, controls gravitation. To understand gravitation requires an understanding of matter in motion. But matter is an aggregate of particles. Thus, to mathematize (and hence to understand) gravitation one must mathematize (an aggregate of) particles in motion, including relativistic motion.

I. Comoving Volume

(8.2)

This process starts by considering an aggregate of particles in a region of spacetime large enough so that one can talk about density, flux, pressure, etc., but then small enough so that these particles can be said to have the same velocity in a given volume element.

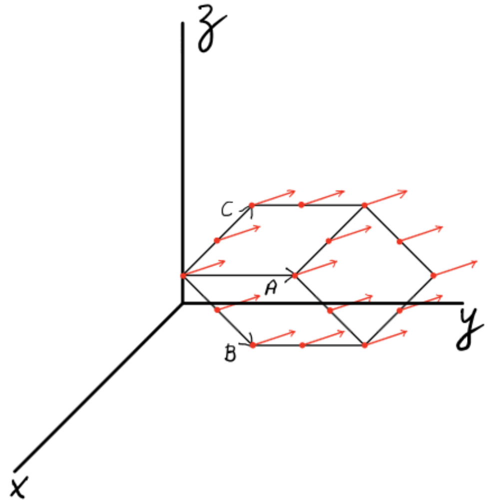

Figure 8.1 A uniform aggregate of particles having the same velocity throughout an element of volume as well as on its boundary.

The common properties of this aggregate is observed

relative to two reference frames, (8.3a)
a LAB frame and the COMOVING frame. The latter is determined entirely by the particles.
The particles have zero spatial velocity in the COMOVING frame $\{e_\mu\} = \{\frac{\partial}{\partial x^\mu}\}$. Their common 4-velocity is

$$\mathbf{u} = u^\mu \frac{\partial}{\partial x^\mu} = 1\frac{\partial}{\partial x^0} + 0\frac{\partial}{\partial x^1} + 0\frac{\partial}{\partial x^2} + 0\frac{\partial}{\partial x^3} \equiv \frac{d}{d\tau}$$

By contrast, relative to the LAB frame their 4-velocity* is

$$\mathbf{u} = u^{\bar\mu} \frac{\partial}{\partial \bar x^{\bar\mu}} = u^{\bar 0}\frac{\partial}{\partial \bar x^{\bar 0}} + u^{\bar 1}\frac{\partial}{\partial \bar x^{\bar 1}} + u^{\bar 2}\frac{\partial}{\partial \bar x^{\bar 2}} + u^{\bar 3}\frac{\partial}{\partial \bar x^{\bar 3}} \equiv \frac{d}{d\tau}$$

The element of 3-volume is spanned by the three space-like vectors

$$A \equiv 0\cdot\frac{\partial}{\partial t} + \Delta x\frac{\partial}{\partial x} + 0\cdot\frac{\partial}{\partial y} + 0\cdot\frac{\partial}{\partial z} \equiv A^\alpha \frac{\partial}{\partial x^\alpha}$$

$$B \equiv 0\cdot\frac{\partial}{\partial t} + 0\cdot\frac{\partial}{\partial x} + \Delta y\cdot\frac{\partial}{\partial y} + 0\cdot\frac{\partial}{\partial z} \equiv B^\alpha \frac{\partial}{\partial x^\alpha}$$

$$C \equiv 0\cdot\frac{\partial}{\partial t} + 0\cdot\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + \Delta z\frac{\partial}{\partial z} \equiv C^\alpha \frac{\partial}{\partial x^\alpha}$$

8.3b

* \footnote{The "bar" over an index is a reminder that the components are relative to the basis

$$\{\bar{e}_{\bar{\mu}}\} = \{\frac{\partial}{\partial \bar{x}^{\bar{\mu}}}\}$$

for the LAB frame. The basis elements without such a bar,

$$\{e_\mu\} = \{\frac{\partial}{\partial x^\mu}\},$$

are for the COMOVING frame.}

As depicted in Figure 8.1, their components (8.4)
are attached to respective pairs of particles in the COMOVING frame with its orthonormal comoving basis $\{e_\mu \frac{\partial}{\partial x^\mu}\}$. Consequently, the element of <u>proper</u> (= comoving) <u>volume</u> is

$$\Delta x\,\Delta y\,\Delta z = \det\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \Delta x & 0 & 0 \\ 0 & 0 & \Delta y & 0 \\ 0 & 0 & 0 & \Delta z \end{vmatrix}$$

$$= \det\begin{vmatrix} U^0 & U^1 & U^2 & U^3 \\ A^0 & A^1 & A^2 & A^3 \\ B^0 & B^1 & B^2 & B^3 \\ C^0 & C^1 & C^2 & C^3 \end{vmatrix}$$

$$= U^\mu \epsilon_{\mu\alpha\beta\gamma} \langle dx^\alpha, A\rangle \langle dx^\beta, B\rangle \langle dx^\gamma, C\rangle$$

$$= U^\mu \epsilon_{\mu\alpha\beta\gamma} \frac{dx^\alpha \wedge dx^\beta \wedge dx^\gamma}{3!}(A,B,C) \quad (8.1)$$

= comoving volume spanned by $\{A, B, C\}$

Comment 8.1

As exhibited by Eq.(8.1), this volume element (which is depicted in Figure 8.2 on page 8.8) is presented in a form which is <u>frame-invariant.</u>

8.5

II. Comoving Particle Density

Consider the density of particles in the volume element $\Delta x \Delta y \Delta z$ spanned by $\{A, B, C\}$,

$$N = \frac{\#}{\Delta x \Delta y \Delta z} = \frac{\begin{pmatrix}\text{number of}\\ \text{particles}\end{pmatrix}}{\begin{pmatrix}\text{comoving}\\ \text{volume}\end{pmatrix}}$$

The number of particles in the comoving volume element is a frame invariant; it is independent of an observer's reference frame. Thus

$$\begin{aligned}\# &= N \Delta x \Delta y \Delta z \\ &= N u^{\mu} \epsilon_{\mu|\alpha\beta\gamma|} dx^{\alpha} \wedge dx^{\beta} \wedge dx^{\gamma}(A, B, C) \\ &= N \bar{u}^{\bar{\mu}} \epsilon_{\bar{\mu}|\bar{\alpha}\bar{\beta}\bar{\gamma}|} d\bar{x}^{\bar{\alpha}} \wedge d\bar{x}^{\bar{\beta}} \wedge d\bar{x}^{\bar{\gamma}}(A, B, C)\end{aligned}$$

Relative to <u>any</u> basis $\{e_{\sigma'}\}$ and its dual $\{\omega^{\sigma}\}$,

$$\omega^{\sigma}(e_{\sigma'}) = \langle \omega^{\sigma}, e_{\sigma'} \rangle = \delta^{\sigma}_{\sigma'},$$

(be it coordinate induced, orthonormal, oblique, etc)

the frame invariance of the particle count is (8.6)
mathematized by the statement

$$\# = N\, u^{\mu'}\, \epsilon_{\mu'|\alpha'\beta'\gamma'|}\, \omega^{\alpha'}\wedge\omega^{\beta'}\wedge\omega^{\gamma'}(A,B,C). \qquad (8.2)$$

This statement says that # depends on the spanning vectors in the manner of a <u>trilinear function.</u>
Given the tensors N, $u = u^{\mu'} e_{\mu'}$, and $\epsilon_{\mu'\alpha'\beta'\gamma'}\, \omega^{\alpha'}\wedge\omega^{\beta'}\wedge\omega^{\gamma'}$ in Eq. (8.2), there must be some chosen triad of vectors (A, B, C) (such as the one on page 8.3), but there may be any such chosen triad.
Using this "some but any" principle* one arrives at the concept $^*S$, the <u>density-flux 3-form.</u>
It is defined by the expression

$$\boxed{N u^{\mu}\, \epsilon_{\mu\alpha\beta\gamma}\, \frac{\omega^{\alpha}\wedge\omega^{\beta}\wedge\omega^{\gamma}}{3!} \equiv {}^*S} \qquad (8.3)$$

* \footnote{This principle was identified and used by Ayn Rand, "Introduction to Objectivist Epistemology", in her theory of concept formation in the chapter "Abstraction From Abstraction". It, among others, deals with the process of widening and narrowing a concept}

(8.7)

More explicitly, the concept of the density-flux 3-form $^*\mathbf{S}$ is a trilinear map, a scalar-valued tensor of rank $\binom{0}{3}$. These and other observations yield the following more explicit

Definition (Density-flux 3 form)

$$^*\mathbf{S}: \quad V \times V \times V \longrightarrow R^1 \quad (\text{\# of particles})$$

$$(A,B,C) \rightsquigarrow {}^*\mathbf{S}(A,B,C) = N\, u^\mu\, \epsilon_{\mu|\alpha\beta\gamma|}\, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma (A,B,C).$$

One of the volume spanning vectors A, B, and C can be time-like. This is highlighted by the qualitative but broader definition

$$^*\mathbf{S} = \frac{(\text{\# of particles})}{(\text{as-yet-unspecified 3-volume})}.$$

It implies that the element of volume, besides being purely space-like, can also have one of its spanning vectors be time-like. Under this circumstance the element of volume is said to be time-like.

Both cases play a key role in the motion of matter from the spacetime perspective.

Case (1)

The spacetime 3-volume is spanned by three spacelike vectors. In the COMOVING frame one has

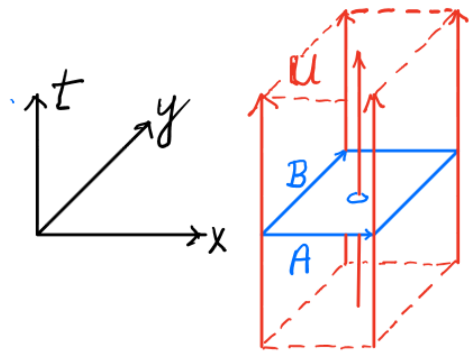

Figure 8.2 Comoving frame representation of finite world tube filled with particle world lines, all with 4-velocity $\mathbf{u}$, passing through the 2-d rendition of the 3-d spatial volume element spanned by A, B and C.

$$\# = {}^{*}\mathcal{S}(A,B,C) = N u^{\mu} \epsilon_{\mu\alpha\beta\gamma} A^{\alpha} B^{\beta} C^{\gamma}$$

$$= \underbrace{N u^{0}}_{S^0}\,\underbrace{\epsilon_{0\alpha\beta\gamma} A^{\alpha} B^{\beta} C^{\gamma}}_{\text{COMOVING volume}}$$

$$S^{0} = \frac{\#}{(\text{COMOVING volume})} = \begin{pmatrix}\text{comoving}\\ \text{particle}\\ \text{density}\end{pmatrix}$$

In the LAB frame consider the triad of space-like vectors

$$\bar{A} \equiv 0\cdot\frac{\partial}{\partial \bar{t}} + \Delta \bar{x}\,\frac{\partial}{\partial \bar{x}} + 0\cdot\frac{\partial}{\partial \bar{y}} + 0\cdot\frac{\partial}{\partial \bar{z}} \equiv \bar{A}^{\bar{\alpha}}\,\frac{\partial}{\partial \bar{x}^{\bar{\alpha}}}$$

$$\bar{B} \equiv 0\cdot\frac{\partial}{\partial\bar{t}} + 0\cdot\frac{\partial}{\partial\bar{x}} + \Delta\bar{y}\,\frac{\partial}{\partial\bar{y}} + 0\cdot\frac{\partial}{\partial\bar{z}} \equiv \bar{B}^{\bar{\alpha}}\frac{\partial}{\partial\bar{x}^{\bar{\alpha}}} \qquad (8.4)$$ 8.9

$$\bar{C} \equiv 0\cdot\frac{\partial}{\partial\bar{t}} + 0\cdot\frac{\partial}{\partial\bar{x}} + 0\frac{\partial}{\partial\bar{y}} + \Delta\bar{z}\,\frac{\partial}{\partial\bar{z}} \equiv \bar{C}^{\bar{\alpha}}\frac{\partial}{\partial\bar{x}^{\bar{\alpha}}}$$

Each one is a spatial displacement connecting a pair of events (a) simultaneous in the LAB and (b) on the boundary of the particle world tube.

Consequently, the number of world lines intercepted by $u^{\mu}\epsilon_{\mu\bar{\alpha}\bar{\beta}\bar{\gamma}}\bar{A}^{\bar{\alpha}}\bar{B}^{\bar{\beta}}\bar{C}^{\bar{\gamma}}$, i.e. the number of particles observed in volume element spanned by $\bar{A}$, $\bar{B}$, and $\bar{C}$, is the same as before, but the components are relative to the LAB basis $\{\bar{e}_{\bar{\mu}} = \frac{\partial}{\partial\bar{x}^{\bar{\mu}}}\}$.

Consequently,

$$\begin{aligned}\# &= N\,u^{\bar{\mu}}\epsilon_{\bar{\mu}\bar{\alpha}\bar{\beta}\bar{\gamma}}\bar{A}^{\bar{\alpha}}\bar{B}^{\bar{\beta}}\bar{C}^{\bar{\gamma}}\\ &= N\,u^{\bar{0}}\epsilon_{\bar{0}\bar{\alpha}\bar{\beta}\bar{\gamma}}\bar{A}^{\bar{\alpha}}\bar{B}^{\bar{\beta}}\bar{C}^{\bar{\gamma}}\\ &= \underbrace{N u^{\bar{0}}}_{S^{\bar{0}}}\underbrace{\epsilon_{\bar{0}\bar{1}\bar{2}\bar{3}}\,\Delta\bar{x}\,\Delta\bar{y}\,\Delta\bar{z}}_{\text{LAB volume}}\end{aligned}$$

$$S^{\bar{0}} = \frac{\#}{(\text{LAB volume})} = \frac{\text{particle density}}{(\text{in LAB frame})} \qquad (8.5)$$

(8.10)

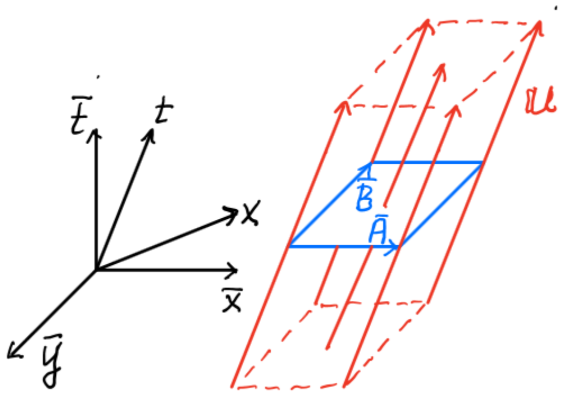

Figure 8.3 World tube filled with straight world lines of type u that pass through the LAB volume element $\Delta\bar{x}\,\Delta\bar{y}\,\Delta\bar{z}$ spanned by space-like vectors $\bar{A}$, $\bar{B}$, and $\bar{C}$.

Case (2)

The spacetime 3-volume is spanned by one time-like vector ($\bar{\bar{A}}$) and two space-like vectors ($\bar{\bar{B}}, \bar{\bar{C}}$). As a linear combination of the LAB basis $\{\bar{e}_{\bar{\mu}}\} = \{\frac{\partial}{\partial \bar{x}^{\bar{\mu}}}\}$ these vectors are

$$
\begin{aligned}
\bar{\bar{A}} &\equiv \Delta\bar{\bar{t}}\cdot\frac{\partial}{\partial\bar{\bar{t}}} + 0\cdot\frac{\partial}{\partial\bar{x}} + 0\frac{\partial}{\partial\bar{y}} + 0\cdot\frac{\partial}{\partial\bar{z}} \equiv \bar{\bar{A}}^{\bar{\alpha}}\frac{\partial}{\partial\bar{x}^{\bar{\alpha}}} \\
\bar{\bar{B}} &\equiv 0\cdot\frac{\partial}{\partial\bar{\bar{t}}} + 0\cdot\frac{\partial}{\partial\bar{x}} + \Delta\bar{\bar{y}}\cdot\frac{\partial}{\partial\bar{y}} + 0\cdot\frac{\partial}{\partial\bar{z}} \equiv \bar{\bar{B}}^{\bar{\alpha}}\frac{\partial}{\partial\bar{x}^{\bar{\alpha}}} \qquad (8.6) \\
\bar{\bar{C}} &\equiv 0\cdot\frac{\partial}{\partial\bar{\bar{t}}} + 0\cdot\frac{\partial}{\partial\bar{x}} + 0\frac{\partial}{\partial\bar{y}} + \Delta\bar{\bar{z}}\,\frac{\partial}{\partial\bar{z}} \equiv \bar{\bar{C}}^{\bar{\alpha}}\frac{\partial}{\partial\bar{x}^{\bar{\alpha}}}
\end{aligned}
$$

8.11

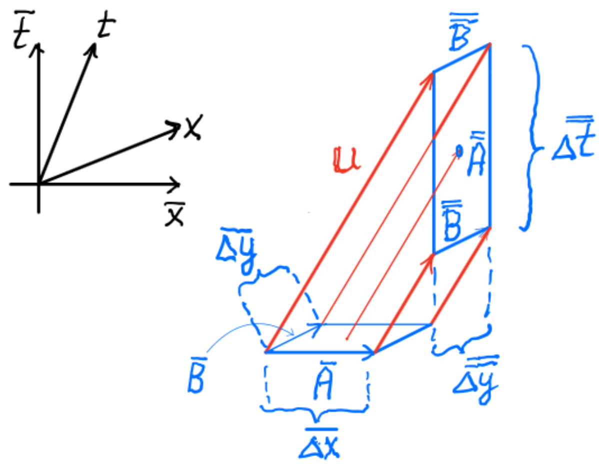

Figure 8.4 The 4-d world tube, which is bounded by 3-d spatial volume element $\overline{\Delta x}\,\overline{\Delta y}\,\overline{\Delta z}$ and by 3-d time-like volume element $\overline{\overline{\Delta t}}\,\overline{\overline{\Delta y}}\,\overline{\overline{\Delta z}}$, is filled with straight world lines of type u. They pass through the LAB volume element $\overline{\Delta x}\,\overline{\Delta y}\,\overline{\Delta z}$ spanned by space-like vectors $\bar{A}$, $\bar{B}$, and $\bar{C}$ and through the time-like volume element spanned by space-like vectors $\bar{\bar{B}}$ and $\bar{\bar{C}}$ and by time-like vector $\bar{\bar{A}}$

The number of particles observed in the time-like 3 volume spanned by $\bar{\bar{A}}$, $\bar{\bar{B}}$, and $\bar{\bar{C}}$ is given by the statement that

$$\# = {}^*S(\bar{\bar{A}}, \bar{\bar{B}}, \bar{\bar{C}}) = u^\mu \epsilon_{\mu\alpha\beta\gamma} \bar{\bar{A}}^\alpha \bar{\bar{B}}^\beta \bar{\bar{C}}^\gamma. \qquad (8.7)$$

(8.12)

This is an example of a statement which is \emph{objective}. This is because it combines #, a metaphysical feature of the world, with a conceptual method of processing the observed data about #, namely by using that data to evaluate the r.h.s. of Eq.(8.6). The act of gathering the observed data and processing it is always the same regardless of the inertial reference frame where the action was done.

Relative to the LAB frame, whose standard of measurements is the LAB basis $\{\bar{e}_{\bar\mu} = \frac{\partial}{\partial \bar{x}^{\bar\mu}}\}$, the evaluation of $^*S$ yields

$$\# = {}^*S(\bar{\bar{A}}, \bar{\bar{B}}, \bar{\bar{C}}) = u^{\bar\mu}\,\epsilon_{\bar\mu|\bar\alpha\bar\beta\bar\gamma|}\,\bar{\bar{A}}^{\bar\alpha}\bar{\bar{B}}^{\bar\beta}\bar{\bar{C}}^{\bar\gamma}$$

$$= \underbrace{N u^{\bar 1}}\,\epsilon_{\bar1\bar0\bar2\bar3}\,\underbrace{\overline{\overline{\Delta t}}}_{\bar{\bar{A}}\text{'s lab time window}}\,\underbrace{\overline{\overline{\Delta y}}\,\overline{\overline{\Delta z}}}_{\text{lab area spanned by } \bar{\bar{B}} \text{ and } \bar{\bar{C}}}$$

$$S^{\bar 1} = \frac{\#}{(\text{time})(\overrightarrow{\text{area}})^1} = \frac{\#}{(\text{time})(\vec{B}\times\vec{C})^1} \qquad (8.8)$$

(8.13)

Summary

1. Based on Eq. (8.5) on page 8.9, the evaluation

$$^{*}\mathbf{S} = S^{\mu}\, \epsilon_{\mu|\alpha\beta\gamma|}\; dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}$$

on the element of volume spanned by $\bar{A}$, $\bar{B}$, and $\bar{C}$ yields

$$S^{0} = \frac{(\text{particles})}{(\text{volume})} = \text{particle density}$$

Based on Eq. (8.8) on page 8.12, the evaluation of $^{*}\mathbf{S}$ on the element of volume spanned by $\bar{\bar{A}}$, $\bar{\bar{B}}$, and $\bar{\bar{C}}$ yields

$$\left.\begin{matrix} S^{1} \\ S^{2} \\ S^{3} \end{matrix}\right\} S^{i} = \frac{\text{particles}}{(\text{time})\,(\overrightarrow{\text{area}})^{i}} = \begin{matrix}\text{particle flux into} \\ \text{the } i^{th} \text{ direction}\end{matrix}$$

2. The vector formed from the proper particle density N and the particle 4-velocity $\mathbf{u}$,

$$N\mathbf{u} \equiv \mathbf{S} = S^{\mu}\mathbf{e}_{\mu}$$

is called the <u>particle current 4-vector.</u>