Lecture 9

The particle density-flux 3 form:

I. Physical properties

II. Algebraic properties
III. Geometrical properties

Consigned to Lecture 10

In MTW know Box 4.4
Box 5.1
Fig 5.1
Box 5.2

The particle density-flux 3-form

$$ {}^{*}\mathcal{S} = \underbrace{N u^\mu}_{\equiv\, S^\mu} \underbrace{\epsilon_{\mu\alpha\beta\gamma} \frac{dx^\alpha \wedge dx^\beta \wedge dx^\gamma}{3!}}_{\Sigma_\mu} \qquad (9.1) $$

mathematizes the physical spacetime properties of a continuous medium of particles. This included the types of media found in extremely relativistic astrophysical environments as well as those driven by ultra-intense laser radiation.

I. Physical Properties of the Particle Density-flux 3-form

The 3-form ${}^{*}\mathcal{S}$ is the result of the mental integration ("unification") of four physical attributes of particles in motion:

(i) their common 4-velocity

$$ \boxed{\mathbf{u} = u^\mu \frac{\partial}{\partial x^\mu} \equiv u^\mu \mathbf{e}_\mu} \qquad (9.2) $$

and their associated particle current

4-vector (9.2)

$$\boxed{N u^\mu e_\mu = S^\mu \frac{\partial}{\partial x^\mu} \equiv \mathbf{S}} \qquad (9.3)$$

(ii) their numerical particle count (particle "number") in the 3-volume spanned by three freely chosen 4-d vectors A, B, and C, is

$$\# = N u^\mu \epsilon_{\mu\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma / 3!\, (A,B,C)$$
$$= N u^\mu \Sigma_\mu (A,B,C).$$

Being a scalar, this is a frame invariant, i.e. independent of the inertial frame for evaluating the scalar.

Comment 9.1 (Calculation)

If all three vectors are space-like, the evaluation yields the particle density.

(a) Relative to the COMOVING instantaneous inertal frame, where $u^\mu \frac{\partial}{\partial x^\mu} = 1\frac{\partial}{\partial x^0} + 0\frac{\partial}{\partial x^1} + 0\frac{\partial}{\partial x^2} + 0\frac{\partial}{\partial x^3}$, the density is obtained from

$$\# = N u^\mu \epsilon_{\mu\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma / 3!\, (A,B,C)$$

$$= N \det \begin{vmatrix} u^0 & 0 & 0 & 0 \\ 0 & A^1 & A^2 & A^3 \\ 0 & B^1 & B^2 & B^3 \\ 0 & C^1 & C^2 & C^3 \end{vmatrix}$$

(9.3)

$$= N u^{\circ}(\vec{A}\cdot\vec{B}\times\vec{C}).$$

Thus,

$$\boxed{S^{0} = N = \frac{\#}{(\vec{A}\cdot\vec{B}\times\vec{C})} = \frac{\#}{\left(\begin{smallmatrix}\text{COMOVING}\\ \text{VOLUME}\end{smallmatrix}\right)} = \left(\begin{smallmatrix}\text{comoving density}\\ \text{of particles}\end{smallmatrix}\right).} \qquad (9.4)$$

(b) Relative to the LAB frame, where

$$u^{\bar\mu}\frac{\partial}{\partial x^{\bar\mu}} = u^{\bar 0}\frac{\partial}{\partial x^{\bar 0}} + u^{\bar 1}\frac{\partial}{\partial x^{\bar 1}} + u^{\bar 2}\frac{\partial}{\partial x^{\bar 2}} + u^{\bar 3}\frac{\partial}{\partial x^{\bar 3}}$$

and the three chosen space-like vector are $\bar{A}$, $\bar{B}$, and $\bar{C}$ with their tips and tails simultaneous in the LAB, the evaluation yields

$$\# = N u^{\bar\mu}\,\epsilon_{\bar\mu\bar\alpha\bar\beta\bar\gamma}\; d\bar{x}^{\bar\alpha}\wedge d\bar{x}^{\bar\beta}\wedge d\bar{x}^{\bar\gamma}/3!\,(\bar{A},\bar{B},\bar{C})$$

$$= N \det\begin{vmatrix} u^{\bar 0} & u^{\bar 1} & u^{\bar 2} & u^{\bar 3} \\ 0 & \bar{A}^{\bar 1} & \bar{A}^{\bar 2} & \bar{A}^{\bar 3} \\ 0 & \bar{B}^{\bar 1} & \bar{B}^{\bar 2} & \bar{B}^{\bar 3} \\ 0 & \bar{C}^{\bar 1} & \bar{C}^{\bar 2} & \bar{C}^{\bar 3} \end{vmatrix}$$

$$= N u^{\bar 0}(\vec{\bar{A}}\cdot\vec{\bar{B}}\times\vec{\bar{C}})$$

$$= N u^{\bar 0}\,(\text{LAB volume}).$$

Thus,

$$\boxed{S^{\bar 0} = N u^{\bar 0} = \frac{\#}{(\text{LAB volume})} \equiv \frac{\text{particle density}}{\text{in LAB frame}}} \qquad (9,5)$$

Comment 9.2

The particle 4-velocity

$$u = u^{\bar\mu}\frac{\partial}{\partial \bar{x}^{\bar\mu}} = u^{\mu}\frac{\partial}{\partial x^{\mu}}$$

is one and the same in both frames. However, in the LAB frame one has

$$\{u^{\bar\mu}\}=\left\{\frac{1}{\sqrt{1-\vec\beta^2}}\ \frac{\vec\beta}{\sqrt{1-\vec\beta^2}}\right\}.$$

9.4

Thus, in the LAB frame

$$S^{\bar 0}\equiv N u^{\bar 0} \geqslant N = N u^{0} \equiv S^{0}.$$

This means that the observed density of moving matter is larger than the comoving density.

Comment 9.3

If one of the three chosen 4-vectors $\bar{\bar A}$, $\bar B$, and $\bar C$ is time like, while the other two are space-like with tip and tail events simultaneous in the LAB frame, then

$$\# = N u^{\bar\mu}\,\epsilon_{\bar\mu\bar\alpha\bar\beta\bar\gamma}\ d\bar x^{\bar\alpha}\wedge d\bar x^{\bar\beta}\wedge d\bar x^{\bar\gamma}/3!\,(\bar{\bar A},\bar B,\bar C)$$

$$= N\ \det\begin{vmatrix} u^{\bar 0} & u^{\bar 1} & u^{\bar 2} & u^{\bar 3}\\ -\Delta\bar t & 0 & 0 & 0\\ 0 & \vec B^{\bar 1} & \vec B^{\bar 2} & \vec B^{\bar 3}\\ 0 & \vec C^{\bar 1} & \vec C^{\bar 2} & \vec C^{\bar 3}\end{vmatrix}$$

$$= N\,\Delta\bar t\left[u^{\bar 1}(\vec B\times\vec C)^{\bar 1}+u^{\bar 2}(\vec B\times\vec C)^{\bar 2}\ \ u^{\bar 3}(\vec B\times\vec C)^{\bar 3}\right]$$

$$= N\,\Delta\bar t\left[\vec u\cdot\vec B\times\vec C\right]$$

This # is the number of particles passing through the area $\vec B\times\vec C$ during the LAB time $\Delta\bar t$. On a per unit time basis one has

$$\boxed{\frac{\#}{\Delta\bar t}=N u^{\bar 1}(\vec B\times\vec C)^{\bar 1}+N u^{\bar 2}(\vec B\times\vec C)^{\bar 2}+N u^{\bar 3}(\vec B\times\vec C)^{\bar 3}=\frac{(\text{particle current})}{(\text{through area } \vec B\times\vec C)}}$$

The three parts contributing to this current are due to the three flux components: (9.5)

$$\boxed{\begin{aligned} S^{\bar i} = N u^{\bar i} &= \frac{\#}{\Delta \bar t \,\left(\vec{\bar B}\times\vec{\bar C}\right)^{\bar i}} = \frac{\text{number of particles}}{\begin{pmatrix}\text{LAB}\\ \text{time}\end{pmatrix}\begin{pmatrix}\text{area whose}\\ \text{normal points into}\\ \text{the } i^{th} \text{ direction}\end{pmatrix}} \\ &= \begin{pmatrix}\bar i^{th} \text{ component}\\ \text{of the particle flux}\end{pmatrix} \qquad \bar i = 1,2,3 \end{aligned}} \tag{9.6}$$

Having identifies the four physical attributes via the boxed Eqs. (9.1)-(9.6), integrate the LAB particle density, Eq. (9.5) and the LAB flux components, Eq. (9.6) into the spacetime framework of the single concept, the <u>particle current 4-vector</u>

$$S = N u = N u^{\bar\mu}\frac{\partial}{\partial x^{\bar\mu}} : \quad \boxed{\begin{aligned} N u^{\bar 0} &= \frac{\text{particles}}{\text{volume}} \\ N u^{\bar 1} &= \frac{\text{particles}}{(\text{time})\,(\text{area})^{\bar 1}} \\ N u^{\bar 2} &= \frac{\text{particles}}{(\text{time})\,(\overrightarrow{\text{area}})^{\bar 2}} \\ N u^{\bar 3} &= \frac{\text{particles}}{(\text{time})\,(\overrightarrow{\text{area}})^{\bar 3}} \end{aligned}}$$