Appendix to Lecture 21:

Chapter 8: The Bianchi Identities, E. Cartan, 1922

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Translation by Thomas Sinha

1 Exterior Differential Forms (see E. Cartan, Lectures on Integral Invariants, 1922)

1.1 181

We consider differential forms with exterior multiplication, (or briefly "exterior differential forms"), which are the forms which occur under the sign in multiple integrals. They obey certain rules of calculation which we will here indicate.

Take for example, in ordinary space of three dimensions, a double integral considered over a portion of the surface:

$$I = \iint P \; dy\, dz + Q \; dx\, dz + R \; dx\, dy$$

In the differential form:

$$m = P \; dy\, dz + Q \; dz\, dx + R \; dx\, dy \;,$$

the terms $dy\, dz$, $dx\, dz \; dx\, dy$ are not ordinary products. If we express the coordinates of a point on the surface of integration as a function of two parameters, α and β, we can regard α and β as the coordinates of a point in another plane; the integral can then be reduced as an ordinary double integral considered on a certain region of this plane. To perform this reduction, we replace respectively the symbols

$$dy\, dz, \; dz\, dx, \; dx\, dy$$

with the quantities:

$$\frac{D(y,z)}{D(\alpha\beta)} d\alpha\, d\beta, \;\; \frac{D(z,x)}{D(\alpha\beta)} d\alpha\, d\beta, \;\; \frac{D(x,y)}{D(\alpha\beta)} d\alpha\, d\beta$$

We see as a result that $dy \; dz$ must not be confused with $dz \; dy$, which must be regarded as equal and opposite of dy dz.

We want to present these things in a suitable manner. We introduce two differentiation symbols d_1 and d_2, and pose that:

$$d_1 u = \frac{\partial u}{\partial \alpha} d\alpha, \;\; d_2 u = \frac{\partial u}{\partial \beta} d\beta$$

These two differentiation symbols commute. With this notation, we have:

$$dy\,dz = \begin{vmatrix} d_1\,y & d_1\,z \\ d_2\,y & d_2\,z \end{vmatrix}, \quad dy\,dz = \begin{vmatrix} d_1\,y & d_1\,z \\ d_2\,y & d_2\,z \end{vmatrix}, \quad dx\,dy = \begin{vmatrix} d_1\,x & d_1\,y \\ d_2\,x & d_2\,y \end{vmatrix}$$

The quantities $dy\,dz$, $dz\,dx$, $dx\,dy$ are products, but with respect to *exterior multiplication* (Grassmann multiplication), the sign of the product changes when we change the order of the factors.

We could more generally introduce any two differentiation symbols, d_1 and d_2, which are interchangeable. If the results of the operations d_1 and d_2 are infinitesimally small, we can decompose the surface of integration in a network of tiny, curvilinear paralellograms, each of which would have respectively:

$$x, y, z$$

$$x + d_1x, y + d_1y, z + d_1z$$

$$x + d_1x + d_2d_1x, y + d_1y + d_2d_1y, z + d_1z + d_2d_1z$$

$$x + d_2x, y + d_2y, z + d_2z;$$

and the integral would become the sum of the quantities:

$$P(d_1y\ d_2z - d_1z\ d_2y) + Q(d_1z\ d_2x - d_1x\ d_2z) + R(d_1x\ d_2y - d_1y\ d_2z)$$

Taken over all of these elementary parallelograms. In fact, these quantities $dy\,dz$, $dz\,dx$, $dx\,dy$ behave like the components of a simple bivector.

We will make it a convention, to avoid any confusion, to put an exterior product in square brackets when such a product is not found under an integration sign (c.f. §**45, 147, 160, 163, 178**). [Aside: we instead, in modern parlance, would use the wedge product].

1.2 182

The previous considerations on multiple integrals are understood in any number of dimensions and lead to sums of terms such as:

$$\Lambda[dx_1\,dx_2\,...dx_p]$$

The exterior product in square brackets is in place of a determinant of order p involving p interchangeable differentiation symbols. Such a product changes sign when one exchanges two of the factors with each other (the coefficient Λ naturally is not considered as a factor from this point of view).

Given two differential forms, ω_1 and ω_2, one of order p and the other of order q, we define the exterior product $[\omega_1\omega_2]$ of these two forms as a p+q form obtained by doing, in any way possible, the exterior product of the first form by the second, and respecting the order in which the differentials are presented. So, for example, suppose we have:

$$\omega_1 = a_i dx^i, \qquad \omega_2 = b_{ij}[dx^i dx^j]$$

We will have:

$$[\omega_1\omega_2] = a_i b_{jk}[dx^i dx^j dx^k]$$

1.3 183

There are a whole series of important formulas that allow us to transform a multiple integral of order p over a closed domain into a multiple integral of order p+1 over a domain which has the first domain as a boundary. The simplest of these formulas, given by Cauchy/Green, is

$$\int Pdx + Qdy = \iint \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dxdy \tag{1}$$

Next comes Stokes' Equation:

$$\int Pdx + Qdy + Rdz = \iint \left(\frac{dR}{dy} - \frac{dQ}{dz}\right) dydz + \left(\frac{dP}{dz} - \frac{dR}{dx}\right) dzdx + \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dxdy \tag{2}$$

Then the Odstrogradsky formula:

$$\iint Pdy\ dz + Qdz\ dx + Rdx\ dy = \iiint \left(\frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz}dx\right) dy\, dz \tag{3}$$

In spaces of more than three dimensions, there exists analogous formulae.

The operation which permits the formation of these formulae can be presented in a very simple form. Take first the case of a simple integral $\int \omega(d)$ over a closed contour (C). Let (S) denote the differential form $d\omega$ under the double integral sign; we call this the exterior derivative of the form omega.

We remark that if the exterior derivative of ω is 0, then omega is an *exact differential.*

To transform, now, a double integral to a triple integral, we introduce in the three dimensional domain of integration, 3 interchangeable differentiation symbols, so we are permitted to decompose it into a network of elementary parallelopipeds. We will demonstrate that the double integral ($\iint \omega$) over the surface of the boundary of these parallelopipeds is equal to

$$d_1\omega(d_2, d_3) - d_2\omega(d_1, d_3) + d_3\omega(d_1, d_2)$$

Let $A\, dx\, dy$ be the terms of ω. We can easily verify that:

$$d_1 \left(A \begin{vmatrix} d_2x & d_3x \\ d_2y & d_3y \end{vmatrix}\right) + d_2 \left(A \begin{vmatrix} d_3x & d_1x \\ d_3y & d_1y \end{vmatrix}\right) + d_3 \left(A \begin{vmatrix} d_1x & d_2x \\ d_1y & d_2y \end{vmatrix}\right) = \begin{vmatrix} d_1A & d_2A & d_3A \\ d_1x & d_2x & d_3x \\ d_1y & d_2y & d_3y) \end{vmatrix}$$

As a result, we have

$$\iint A\, dx\, dy = \iiint dA\, dx\, dy$$

In general, if

$$\omega = A_{ij}\, dx^i\, dx^j$$

We have:

$$\iint \omega = \iiint dA_{ij}\, dx_i\, dx_j = \iiint d\omega$$

$d\omega$ designates the exterior derivative of the form ω.

The process of exterior differentiation indicated for forms of order 1 and 2 extends to forms of any order: the exterior derivative of the form:

$$\omega = A[dx_1\, dx_2\, ...dx_p]$$

is

$$d\omega = [dA\, dx_1\, dx_2\, ...dx_p]$$

It is a convention that we can do this, which will be furthered at the end of (**185**).

1.4 184

The integrals which appear in the formulas of Green, Stokes, and Ostrogradsky, are cast in a general form in the relation:

$$\int \omega = \int d\omega$$

Where ω is a p-form, and the second term is considered in a domain of the space, D, of p+1 dimensions, the first is a closed set V in p dimensions which limits this domain. This doesn't make sense if D is not an oriented domain of which the closed set V is the limit. We can orient the domain in convention that a certain (p+1)-hedron formed by p+1 independent vectors $\{e_1, e_2, ...e_{p+1}\}$ from a fixed point is directed; the set V is oriented in a coherent manner by sending each point M in V to a vector e' outside of the domain D and p vectors $\{e'_1, e'_2, ...e'_3\}$ tangent to V such that the (p+1)-hedron formed by $\{e', e'_1, e'_2, ...e'_p\}$ is directed; then the orientation of V will then extend from the p-hedron formed by the vectors $\{e'_1, e'_2, ...e'_p\}$. One can easily verify Greens' formula (and Stokes' as a generalization) is valid with the preceeding conventions, the same holds for Ostrogradsky's. For example, in the case of Green's formula, if one orients the area of the xy-plane which extends to the double integral counterclockwise, the the integral $\int Pdx + Qdy$ over a contour, simple or multiple, from the area which is in the direction of an observer who is looking at it from the left.

1.5 185

The exterior differentiation has certain very simple properties. Take ω to be any differential exterior form, $d\omega$ is its exterior derivative, and m is a given scalar function of the variables. We have:

$$d(m\omega) = m\ d\omega + [dm\ \omega] \tag{4}$$

In fact, if we take omega to be any term such as:

$$A[dx_1...dx_p]$$

Then this corresponds with $m\omega$ as

$$mA[dx_1...dx + p]$$

Then obviously the exterior derivative is

$$m[dAdx_1...dx_p] + A[dmdx_1...dx_p]$$

The addition of all analogous terms demonstrates the theorem. A more general formula is the following. Take ω_1 and ω_2 as two exterior differential forms, with respective order p and q. We consider the form $[\omega_1\omega_2]$ of order p+q. Let

$$A[dx_1...dx_p], \qquad B[dy_1...dy_q]$$

Be ω_1 and ω_2, respectively. Then the corresponding term $[\omega_1\omega_2]$ is

$$AB[dx_1...dx_p\ dy_1...dy_q]$$

The exterior derivative of this is:

$$B[dA\ dx_1...dx_p\ dy_1...dy_q] + A[dB\ dx_1...dx_p\ dy_1...dy_q]$$

The second term can be written as:

$$(-1)^p A[dx_1...dx_p \; dBdy_1...dy_q]$$

Which results in the formula:

$$d[\omega_1\omega_2] = [d\omega_1\omega_2] + (-1)^p[\omega_1 d\omega_2] \tag{5}$$

which generalizes for the product of any number of factors. This generalizes the normal rule for differentiating a product.

We can now demonstrate the operation of exterior derivation, as it has been defined in the general case at the end of 183, is covariant for all changes of variables, in the sense that if we switch the variables x_i with the variablesy_i, and if for this change of variables the form $\omega(x, dx)$ transforms to $\psi(y, dy)$, the exterior derivative $d\omega$ calculated with respect to independent variables x_i transforms to $d\psi$ calculated with respect to the independent variables y_i. In effect, the term $A[dx_1...dx_p]$, considered using the variables x_i, has exterior derivative equal to $[dAdx_1dx_2...dx_p]$, according to the product rule of differentiation, and because the factors $dx_1 \; dx_2...dx_p$ are exact differentials, so their exterior derivative is 0 (183).

1.6 186

There is an important theorem, from Henri Poincare, on successive exterior derivatives of any differential form. The second exterior derivative is exactly 0. In the case where we're in ordinary space and where we start with a linear form:

$$m = Pdx + Qdy + Rdz$$

The theorem is evident. Consider as an example any closed surface (S) and the integral $\iint d\omega$ extended over this closed surface. Lets split up this surface into two parts, (S1) and (S2) by a closed contour (c). The integral $\iint d\omega$ over (S1) is equal to the integral $\int \omega$ on the curve (C), integrated in one direction, and the integral $\iint d\omega$ over (S2) is equal to the integral $\int \omega$ on the curve C *integrated in the opposite direction*. The result is that the integral over the whole surface must be 0, regardless of the surface considered. The exterior derivative $d\omega$ is then identically 0.

The analytic demonstration of this is quite simple. Take

$$A[dx_1dx_2...dx_p]$$

To be the given ω; the term corresponding to $d\omega$ is then:

$$[dAdx_1...dx_p]$$

To take the exterior derivative of this form, we need to can regard it as the exterior product of p+1 factors, each of which is an exact differential: The exterior derivative of this product is therefore null.

This theorem of Poincare's admits a reciprocal, but we will not use it.

2 Tensorial Differential Forms

2.1 187

Besides scalar differential forms, as we have considered so far, it is necessary to consider tensorial differential forms. Lets start in Euclidean space with a fixed cartesian coordinate system. Consider an integration domain in p dimensions, at each point of which we attach an infinitesimally small tensor, the components of which are taken to be a differential form of degree p. If, for example, the tensor is mixed in two indices, each of these components is a differential form ${\omega_i}^j$. The geometrical sum of all of these infinitesimal tensors becomes a tensor of the same form as the components of $\int {\omega_i}^j$. We will be able to exterior differentiate this tensorial differential form, the exterior derivative of ${\omega_i}^j$ becomes $d{\omega_i}^j$

2.2 188

If the euclidean space is given any curvilinear coordinates, there exists a Cartesian representation of each point in the space. To obtain the absolute exterior derivative of a contravariant vectorial form ω^i, we can introduce a uniform vector field with components X_i and consider the sum $X_i\omega^i$; the exterior derivative of such a sum, which is a scalar, is:

$$X_i d\omega^i + dX_k\omega^k = X_i(d\omega^i + [\omega^i_k\omega^k])$$

The absolute exterior derivative is found to be:

$$D\omega^l = d\omega^l + [\omega^l_k\omega^k] \tag{6}$$

We obtain the analogous expression

$$D\omega_l = d\omega_l - [\omega^k_l\omega_l] \tag{7}$$

And, more generally, for a tensorial form with two indices:

$$D{\omega_i}^j = d{\omega_i}^j - {\omega_l}^k\omega^j_k + \omega^j_k\omega^k_l \tag{8}$$

We note that, in the case of ordinary tensors, the absolute derivative of a product is obtained by applying the product rule, but we replace ordinary differentiaion with absolute differentiaion.

We have, for example:

$$D[{a_i}^j{}_k du^k] = [D{a_i}^j{}_k du^k], D[\omega^i y_l] = [D\omega^i y_l] + (-1)^p[\omega^i Dy_l]$$

Where p is the degree of the form ω^i

2.3 189

Now lets place ourselves in a Riemannian Manifold. If we take any integration domain in this space, the geometric sum of infinitely many infinitesimal tensors (for example, vectors) at each point of the domain does not make sense. But if the domain of integration is the entire infinitesimal neighborhood of the point A in the Riemann space, we can substitute a linear element of the Riemann space with a linear element of the Euclidean space resembling A, then the tensorial integral makes sense, and the integrand, which is a tensor at the point A, is independent of the Euclidean metric we chose.

In particular, suppose we have a p+1 dimensional integration domain which has a p-dimensional boundary. The tensorial integral of the element $d{\omega_i}^j$ over this boundary is equal to the integral of the element $D{\omega_i}^j$ over the given domain;, or, at the point A, the coefficients of $D{\omega_i}^j$ do not involve the euclidean metric, but rather the coefficients ${\Gamma_i}^j{}_k$, which are the same as in the Riemannian metric. So we can define a tensorial integral over a infinitesimal domain on a Riemannian manifold, and the operation of absolute exterior differentiation is done according to the same laws as Euclidean space. The previous related theorems on the exterior derivative of a product extend also to Riemannian Manifolds.

2.4 190

Take for example the vectorial integral integral dM considered on a very small loop. We have here that:

$$\omega^i = du^i$$

$$D\omega^i = [\omega^i_k du^k] = -\frac{1}{2}\left(\Gamma_h{}^i{}_k - \Gamma_k{}^i{}_h\right)[du^h du^k] = 0$$

As a result, the geometric sum of vectors $M\vec{M}'$ which join a point of the cycle to a point in the neighborhood is 0. This result can be related to the previous chapters considerations. In effect, it proves if we develop a cycle in Euclidean space, the geometric sum of vectors $M\vec{M}'$ is 0, so, as a result, the displacement associated with a infinitesimal cycle of any form reduces to a rotation.

3 Bianchi Identities

3.1 191

By the formulas given in (160) and (163), we are given the forms $\Omega_i{}^j$ or Ω_{ij} which define the Riemann Curvature:

$$\Omega_i{}^j = d\omega_i{}^j - [\omega_i{}^k \omega_k{}^j] \tag{9}$$

$$\Omega_{ij} = d\omega_{ij} - [\omega_{ik}\omega_j{}^k] \tag{10}$$

We take the exterior derivative of both terms in equation 9; we obtain, taking into account the equation itself, the new relation:

$$d\Omega_i{}^j = -[\Omega_i^k \omega_k^j] + [\omega_i^k \Omega_k^j] \tag{11}$$

If we go back to equation 8, we see that the equation 11 implies that the absolute exterior derivative of the differential tensorial form $\Omega_i{}^j$ is 0, so we write:

$$D\Omega_i{}^j = 0 \tag{12}$$

As the form $\Omega_i{}^j$ is of 2nd degree, $D\Omega_i{}^j$ is of degree 3, and the relations (12) can be expressed, with the notation of absolute differential calculus, in the form:

$$R_i{}^j{}_{\alpha\beta|\gamma} + R_i{}^j{}_{\beta\gamma|\alpha} + R_i{}^j{}_{\gamma\alpha|\beta} = 0 \qquad (i, j, \alpha, \beta, \gamma = 1, 2, ...n) \tag{13}$$

These relations, which only translate the equations (12), constitute what we call the Bianchi Identities.

The tensorial form Ω_{ij}, being simply the form $\Omega_i{}^j$ written in covariant form, also has absolute exterior derivative 0, which gives the identities:

$$R_{ij\alpha\beta|\gamma} + R_{ij\gamma\alpha|\beta} + R_{ij\beta\gamma|\alpha} = 0 \tag{14}$$

Which can be deduced directly from equation (13).

The tensor Ω_{ij}, or rather the negative tensor $-\Omega_{ij}$, represents the bivector which defines the rotation associated with an element of the surface of the space. We deduce immediately from this, and from (189), the geometric significance of these Bianchi Identities:

If one considers an elementary domain in three dimensions of space, the bivectors which represent the rotations associated with elements of the surface which limits the volume, have geometric sum 0.

4 Poincare's Theorem on Riemannian Manifolds

4.1 192

We have seen in (186) that the second exterior derivative of a form is exactly 0; this is the theorem of Poincare. This theorem is obvious in Euclidean space, on any differential tensorial form. It's not the same in general on a Riemannian Manifold. Take, to fix this idea, a vectorial differential form with components ω^i. The absolute exterior derivative (188) is:

$$D\omega^i = d\omega^i + [\omega_k{}^i \omega^k]$$

Take another absolute exterior derivative:

$$D^2\omega^i = d(D\omega^i) + [\omega_k{}^i D\omega^k]$$

This calculation immediately gives us:

$$D^2\omega^i = [\Omega_k{}^i \omega^k] \tag{15}$$

We see here we've introduced the Riemann curvature of the space, which prevents in general the second absolute exterior derivative of ω^i from being 0.

The following exterior derivatives then give:

$$D^3\omega^i = [\Omega_k{}^i D\omega^k]$$

$$D^4\omega^i = [\Omega_k{}^h \Omega_h{}^i om^k]$$

We have analogous expressions starting with any tensorial form. If in particular, if $\omega^i = du^i$, the absolute exterior derivative $D\omega^i$ is is 0, the second derivative is likewise also 0, and as a result, we have from (15):

$$[du^k \Omega_k{}^i] = 0 \tag{16}$$

This relation restores Equation (13), which restricts the components $R_k{}^i{}_{hl}$, $R_h{}^i{}_{lk}$, $R_l{}^i{}_{kh}$ of the curvature tensor; in this sense it may be regarded as a demonstration of Equation (13).

Another interesting application of (15) is obtained by taking for ω^i a field of ordinary contravariant vectors X^i. Consider a cycle (C) limiting an infinitesimal area, that is to say, all of the points are infinite neighborhoods of a point A. The integral $\int DX^i$ over this cycle is equal to the double integral $\iint X^k Om_k{}^i$, considered over the area. If the area is equivalent to an infinitesimal bivector p^{ij}, we have the relation:

$$\int_{(C)} DX^\ell = \frac{1}{2} R_k{}^l{}_{rs} p^{rs} X^k$$

From remark 2 from (158), we can deduce that the geometric variation ∇X^i due to the rotation associated with the cycle is

$$\nabla X^i = -\frac{1}{2} X^k R_k{}^i{}_{rs} p^{rs}$$

The mixed components $a_i{}^j$ of a bivector which represents this rotation are therefore

$$a_i{}^j = -\frac{1}{2} R_i{}^j{}_{rs} p^{rs}$$

This result is identical to the one found in formula 6 in (162), demonstrated in the particular case where the cycle is an infinitesimal parallelogram. We now observe that this result is valid for all infinitesimal cycles, regardless of its form.

Remark: In the formula which gives the covariant exterior derivative of a tensorial form, we deduce that the derivative is 0 if the bivectorial form of the forth degree $[\Omega_i{}^k \Omega_k{}^j]$ is 0. This will be trivial if the space is in 2 or 3 dimensions. One can easily verify that this will still hold if the space is any number of dimensions, and has constant curvature.

5 The Vectorial Curvatures and Their 1st Representation

5.1 193

Returning to the geometric interpretation of the Bianchi Identities. They express (191) that if one considers an element in three-dimensional space, the geometric sum of the bivectors which represent rotations associated with the elements of the surface of the boundary of the domain are 0.

The bivectors in the above statement are free bivectors. Lets see what happens if we consider applied bivectors (19). For each element of the surface there is an associated applied bivector.

$$\frac{1}{2}[Me_ie_j]\Omega^{ij}$$

The geometric sum of all of these applied bivectors is a free trivector; the first is null by the Bianchi identites, so there remains only the free trivector. The integral

$$\iint \frac{1}{2}[Me_ie_j]\Omega^{ij}$$

evidentally gives, by absolute exterior differentiation, the free trivector:

$$\iiint \frac{1}{6}\left(du^i\Omega^{jk} + du^j\Omega^{ki} + du^k\Omega^{ij}\right)[e_ie_je_k]$$

We will agree to say the trivector with components:

$$\Omega^{ijk} = [du^i\Omega^{jk}] + [du^j\Omega^{ki}] + [du^k\Omega^{ij}] \tag{17}$$

Or rather, its negative, represents the trivectorial curvature at the 3-dimensional point considered. The tensor provided with these coefficients has six indices, with

$$R^{ijk}_{ijk} = R^{jk}_{jk} + R^{kl}_{kl} + R^{ij}_{ij}$$

$$R^{ijk}_{ijh} = R^{jk}_{jh} + R^{ik}_{ih}$$

$$R^{ijk}_{ihl} = R^{jk}_{hl}$$

$$R^{ijk}_{hlm} = 0 \qquad (i, j, k, h, l, m \quad \text{distinct})$$

In these formulas, there is not a sum over twice repeated indices, they instead have their values fixed.

5.2 194

Consider now ($n \geq 4$) an elementary domain in 4 dimensions of space and the free trivectorial curvature of the elements of its three- dimensional boundary. Their geometric sum will be given by the absolute exterior derivative of the form Ω^{ijk}; It is null, because the forms du^i and Ω^{jk} have 0 derivative. So the geometric sum of the free curvature trivectors of a boundary of an infinitesimally small domain in 4-dimensions is 0.

If we instead consider the *applied* curvature trivectors, it is no longer the same, and we obtain a quadrivector with components:

$$\begin{aligned} \Omega^{ijkh} &= [du^i\Omega^{jkh}] - [du^j\Omega^{ikh}] + [du^k\Omega^{ijh}] - [du^h\Omega^{ijk}] \\ &= 2\{[du^i du^j\Omega^{kh}] + [du^j du^k\Omega^{ih}] + [du^k du^i\Omega^{jh}] + [du^i du^h\Omega^{jk}] + [du^j du^h\Omega^{ki}] + [du^k du^h\Omega^{ij}]\} \end{aligned} \tag{18}$$

This quadrivector, or rather half of its negative, may be regarded as defining the free quadrivectorial curvature of an element in four dimensions of space.

We can see how we can continue these operations and gradually define the (applied or free) p-vectorial curvature in a p-dimensional space. We have the following theorem:

Theorem: Given an infinitesimal domain in a p-dimensional Riemannian manifold, the geometric sum of the free (p-1) vectorial curvatures on the boundary are 0; the geometric sum of the applied (p-1) vectorial curvatures of the same elements is equal, up to a numerical factor, to the free vectorial curvature on the domain.

5.3 195

Notice in particular what happens for an infinitesimal domain in n-1 dimensions. The n-1 vectorial curvature has components:

$$\Omega^{i_1 i_2 \ldots i_{n-1}} = [du^i_1 \ldots du^i_{n-2}\Omega^{i_{n-2}i_{n-1}}] + \ldots.$$

We have here

$$R^{i_1 i_2 \ldots i_{n-1}}_{i_1 i_2 \ldots i_{n-1}} = \frac{1}{2} R^{i_\alpha i_\beta}_{i_\alpha i_\beta}$$

$$R^{i_1 i_2 \ldots i_{n-2} i_{n-1}}_{i_1 i_2 \ldots i_{n-2} i_n} = R^{k i_{n-1}}_{k i_n}$$

In the left hand side of the above, there is not a sum on repeated indices; on the right-hand side however, i_α and i_β in the first equation are summed over all values of α, β up to n. In the second equation, on the right-hand side, k takes the values 1,2,...n.

We orient the space and designate by $l_i d\sigma$ the covariant components of an vector additional to the n-1 dimensions considered. Likewise, designate $q_i d\sigma$ as the vector additional to the (n-1) vectorial curvature of the given point.

We have, by denoting $R = R^{kh}{}_{kh}$

$$q_i = \frac{1}{2} l_i R - R_{ik} l^k \tag{19}$$

It's introduced as well as the Riemannian curvature scalar R and the contracted curvature tensor R_{ij} (180). These formulae can be interpreted in the following manner.

Consider, in the tangent space at a point in the manifold, the quadric having the point at its center with the equation:

$$S_{ij} X^i X^j = \frac{1}{2} R g_{ij} X^i X^j - R_{ij} X^i X^j = 1$$

We call this the Einstein quadric. The curvature of the element in (n-1) dimensions of magnitude $d\sigma$ may be represented by a vector $q_i d\sigma$ with:

$$q_i = S_{ik} l^k \tag{20}$$

The l^ks denote the contravariant components of a unit vector normal to the given element. We see that the vector is normal in the hyperplane diametrically conjugate to the direction l^i with respect to the Einstein quadric. The general theorem (194) we interpret as the geometric sum of vectors which represent the curvature of the elements of a boundray of an infinitesimal domain in n-dimensions is 0. Analytically, the theorem can be expressed in writing as the divergence of the tensor S_{ij} is 0 or:

$$S_i{}^k{}_{|k} = 0$$

These are, for n=4, the equations which in Einstein's theory express the theorem of the conservation of momentum and energy. The vector which represents the curvature of an element in 3-dimensions of space (space-time) represents nothing else, in fact, than the momentum and energy contained within that element.

We remark that the formulas (19) give, as a specific case, the formulas (16) found to represent the curvature of a 3-dimensional space.

5.4 196

The principle directions of Ricci (180) are at the same time the principle directions of a Ricci Cone and of the Einstein quadric. It is easy now to demonstrate a general theorem of Ricci and that we have already considered in the case of 3 dimensions (171).

Imagine a total geodesic variety V_{n-1}. The normal of this variety remains normal when it is displaced parallel along any path traced in the variety; as a result, the rotation associated with any cycle in the variety leave this normal fixed. The bivector associated with such a cycle is entirely tangent to the variety. It follows immediately that the trivector associated with an element in three dimensions of the variety is itself also entirely tangent to V_{n-1}, since it is a sum of simple tangent trivectors. The reasoning extends the same to an element of V_{n-1} in any number of dimensions. In particular, the (n-1)-vector which represents the curvature of an element in n-1 dimensions of V_{n-1} is tangent to V_{n-1}, and supplementary vector $q_i d\sigma$ is normal in V_{n-1}, that is to say, normal to the element. The normal is therefore a principal direction of the Einstein quadric, that is to say, of the space.

6 The Vectorial Curvatures and Their 2nd Representation

6.1 197

We have in the previous section, defined the Riemann curvature of an element in p-dimensionial space and represented said curvature by a p-vector. There is a second way of representing this by means of an additional (n-p)-vector, which we cannot take to be free or applied. This second representation supposes a pre-existing orientation of the space. We have already used this for p = 3 (169) and p = n-1 (193).

If we take free (n-p)-vectors, the theorem given in (194) is expressed as the sum of free (n-p)-vectors which represent the curvature of the elements of a boundary of a infinitesimal domain in p+1 dimensions is 0. It is very remarkable that, contrary to what we found in the previous section, the geometric sum of the same *applied* (n-p)-vectors is also 0. It will suffice to demonstrate this for p = 4; n = 7.

Take $\Theta^{123} = \frac{1}{\sqrt{-g}}\Omega_{4567}$, the components of a trivector attached to an element in 4-dimensions of space. The free 4-vector represents the geometric sum of applied trivectors on a small domain of 5 dimensions with components Θ^{1234}. In the expression

$$\Theta^{1234} = [du^1\Theta^{234}] - [du^2\Theta^{134}] + [du^3\Theta^{124}] - [du^4\Theta^{123}] = -\frac{1}{\sqrt{-g}}[du^k\Omega_{k567}]$$

Each of the terms which compose Ω_{k567} has a factor of the form $\omega_k = g_{kh}du^h$, or of the form $\Omega_{ki}\quad(i = 5, 6, 7)$, or the sum

$$du^k\omega_k = g_{kh}[du^k du^h]$$

is nulle, so that the sum

$$[du^k\Omega_{ki}]$$

iS 0 by equation (16). This proves that the different components of Θ^{ijkh} are all 0. QED.

6.2 198

In the particular case of p = n-1, the previous theorem expresses that the vectors which represent the curvature of elements of a boundary of an infinitesimal domain in n-dimensions can be regarded as a system of forces in equilibrium.

For n = 4, this theorem completes the physical interpretation of Einstein's gravitational equations: The vectors which represent in mechanics the quantity of "momenergy" are in effect the applied vectors and not the free vectors.

For n = 3, the theorem can take a remarkable mechanical form.

Take A to be a point in a 3-dimensional Riemannian manifold. Attach to this point a rectangular coordinate patch and consider the small domain around the point A. The components $pd\sigma$, $qd\sigma$ and $rd\sigma$ of a vector attached to an element of surface of the boundary of the domain have the form

$$p = K_{11}\alpha + K_{12}\beta + K_{13}\gamma$$

$$q = K_{21}\alpha + K_{22}\beta + K_{23}\gamma$$

$$r = K_{31}\alpha + K_{32}\beta + K_{33}\gamma$$

Where α, β and γ designate the guiding cosines of the normal of the element. These formulas are identical to the ones expressing the elastic forces in a continuous medium. We have, therefore, the following theorem:

If one imagines a Riemannian Manifold in three dimensions like a continuous medium then elastic pressure that is exerted on each element of the surface is equal to a vector which represents the Riemann curvature of that element, and this medium is in equilibrium under the action of elastic forces.

7 F. Schur's Theorem

7.1 199

The previous considerations lead us to naturally what becomes of these theorems relative to the vectorial curvature for an space which is isotropic at each of its points.

In this case of rotation associated with an element of the surface is reduced to a bivector tangent to that element and equal to the product of that element by a scalar K. The contravariant components of this bivector are:

$$-\Omega^{ij} = \mathrm{K}[du^i du^j]$$

This has 0 absolute exterior derivative, and by noticing that the tensor $[du^i du^j]$ is itself 0 (190), we obtain:

$$[dK du^i du^j] = 0$$

If $n \geq 3$, all of the derivatives $\frac{\partial \mathrm{K}}{\partial u^k}$ are 0, therefore K is constant. We have the following theorem, from F. Schur

Theorem: If a Riemannian Manifold of $n \geq 3$ dimensions is isotropic at each of its points, it has constant curvature.

7.2 200

There exists a more general theorem for $n = 4$, from G. Herglotz, on spaces whose principle directions are completely indeterminate, that is to say, for those where the Einstein Quadric is a hypersphere. These spaces are still characterized by the constancy, at each point, of the curvature in different directions *in n-1 dimensions*.

For a space with this property, the (n-1)-vectorial curvature of an element in n-1 dimensions is represented by an (n-1)-vector situated in the same n-1-plane of that element and proportional to that element; the contravariant components are of the form:

$$\Omega^{i_1 i_2 \ldots i_n} = H[du_1^i du_2^i \ldots du_{n-1}^i]$$

The absoulute exterior derivative of the tensorial form is found to 0, and as a result, H is a scalar:

$$[dH du_1^i du_2^i \ldots . du_{n-1}^i] = 0$$

Where if

$$\frac{\partial H}{\partial u_\alpha^i} = 0$$

The curvature H is thereby constant. We have moreover

$$S_{ij} = H g_{ij} = \frac{1}{2} R g_{ij} - R_{ij}$$

Where:

$$R^i{}_j = 0 \quad (i \neq j)$$

$$R^i{}_i = R^{ik}{}_{ik} = \frac{1}{2} R - H$$

Where the index i in the last formula is not a summing index. If we now sum over i, we obtain

$$R = n\left(\frac{1}{2} R - H\right)$$

$$H = \frac{n-2}{2n} R$$

The Riemannian curvature scalar is therefore constant.

The preceeding theorem reduces, for n = 3, to the theorem of Schur. It is to be compared to the Hydrostatic Theorem according to which a perfect fluid in equilibrium under the action of the only elastic forces behaves, under constant pressure.

We'll end with an interesting remark. If the curvature of a Riemannian Manifold is 0 in p dimensions at a point, the Riemann-Christoffel tensor at that point has all zeros as its components. An exception is made if p = n-1, the hypothesis leads to $n(n+1)/2$ relations:

$$R_{ij} = 0,$$

Which expresses that the contracted curvature tensor is 0. The spaces for which these relations are always verified have 0 curvature, but only in the n-1 directions in n dimensions. That's whats happening in Einstein's theory for an empty spacetime, where there is no momentum nor any energy. We give the name Einstein Spaces for spaces for which $R_{ij} = 0$.