Lecture 19

The Gravitational Field Equations:
Einstein versus Cartan

I. Einstein's tensorial line of reasoning

II. Cartan's, Misner, and Wheeler's geometrization of the E.F. Eq'ns

III. "Rotation" as a tensor

IV. Curvature as rotation

I. Einstein's line of reasoning that led to his gravitational field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu},$$

or equivalently

$$R_{\mu\nu} = \frac{8\pi G}{c^2}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right),$$

was a multi step tour de force:

(i) Geometrize Newton's 1st Law relative to non-inertial reference frame

$$\frac{d^2 x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}$$

(ii) Special Relativity:

Uniformly accelerated frame as a sequence of inertial frames.

(iii) Mathematize the dynamical laws governing particles and fields into coordinate frame independent form.

(iv) Recognize and incorporate the Equivalence Principle as the metaphysical corner-
(Here "metaphysical" means: that which pertains to reality, to the nature of things, to existence.)
stone in conceptualizing gravitation:

a) "uniformly acc'd frame $\simeq$ static, homogeneous gravitational field"

or

b) "inertial force = grav'l force"

(v) Apply the Equivalence Principle (E.P.) to the motion of bodies:

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \rightarrow \frac{d^2x^i}{d\tau^2} = -\Gamma^i_{00}$$

$$= \frac{1}{2}g_{00,i} = \begin{pmatrix}\text{"inertial} \\ \text{force"}\end{pmatrix}$$

$$\underset{\text{E.P.}}{=} -\phi_{,i} = \begin{pmatrix}\text{"gravitational} \\ \text{force"}\end{pmatrix}$$

(vi) Mathematize the momenergy properties and the dynamics of matter, particles, and fields in geometrical form based on the momenergy tensor $\{T^{\mu\nu}\}$:

$$T^{\mu\nu}{}_{;\nu} = 0 .$$

(vii) Generalize the Newtonian gravitational field equation

$$\nabla^2 \phi = 4\pi G \rho$$

by taking advantage of

a) the special relativistic mass-energy relation and

b) the fact that the Riemann curvature tensor

$$\{R^\alpha{}_{\beta\gamma\delta}\} = \{\Gamma^\alpha{}_{\beta\gamma,\delta} - \Gamma^\alpha{}_{\beta\delta,\gamma} + \Gamma^\alpha_{\sigma\gamma}\Gamma^\sigma_{\beta\delta} - \Gamma^\alpha_{\sigma\delta}\Gamma^\sigma_{\beta\gamma}\}$$

is the only tensor containing

2^{nd} derivatives of $g_{\mu\nu}$, including
$g_{00}{}^{;i}{}_{;i} = \left(-1-\frac{2\Phi}{c^2}\right)^{;i}{}_{;i} = -2\nabla^2\frac{\Phi}{c^2}$, which
imply that the tensorial generalization of the Newtonian gravitational field equation is

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu} = \text{expression in } T_{\mu\nu} \text{ and } g_{\mu\nu}\cdot T^{\alpha}_{\alpha}$$

(viii) By demanding that momenergy conservation

$$T^{\mu\nu}{}_{;\nu} = 0$$

be contained in a tensorial way of the tensorially generalized Newtonian equations

$$-\nabla^2(\underbrace{g_{00}}_{\left(-1-2\frac{\Phi}{c^2}\right)}) = \frac{8\pi G}{c^2}\rho\,,$$

i.e. $\nabla^2\frac{\Phi}{c^2} = 4\pi G\frac{\rho}{c^2}$, Einstein arrived at

$$R_{\mu\nu} = \frac{8\pi G}{c^2}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\alpha}{}_{\alpha}\right)$$

which is equivalent to

$$\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\equiv G_{\mu\nu}} = \frac{8\pi G}{c^2} T_{\mu\nu}.$$

This equation incorporates mom-energy conservation

$$G_{\mu}{}^{\nu}{}_{;\nu} = 0$$

identically, and has the Newtonian grav'l equations as an asymptotic limit.

COMMENT:

Such a construction and line of reasoning is necessary, but not enough.

In physics and mathematics both sides, the l.h.s. and the r.h.s. of an equation (e.g. a stress-strain relation, $\vec{F} = m\vec{a}$, etc) must have a well-defined identity.

The r.h.s. of Einstein's equation, $T_{\mu\nu}$, is well-defined geometrically and physically.

However, this is not the case for the l.h.s.

II. In 1928 Cartan, and in 1964, 1972, 1990 Misner and Wheeler filled that cognitive gap by restating Einstein's field equation in geometrical form, both for the l.h.s. and the r.h.s.

$$\underbrace{R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R}_{\text{l.h.s}} = \frac{8\pi G}{c^2} \underbrace{T^{\mu\nu}}$$

$$\text{l.h.s} = \underline{\overset{\text{sum of}}{\text{moments}}} \text{ of} \left\{ \begin{array}{c} \text{curvature} \\ \text{induced} \\ \underline{\text{rotation}} \\ \text{for the 6} \\ \text{faces of} \\ \text{a small} \\ \text{3-cube} \end{array} \right\} = \text{r.h.s} = \frac{8\pi G}{c^2} \left\{ \begin{array}{c} \text{amount of} \\ \text{momenergy} \\ \text{inside this} \\ \text{3-cube} \end{array} \right\}$$

A prerequisite for understanding and using the Einstein field equations is that one grasp the meaning and the geometrical formulation of the concepts

(i) "rotation" and (ii) "moment".

III.) ROTATION AS A TENSOR

The Physical Origin of Rotation.

In three dimensions consider a vector $\vec{v}$ rotating with a given angular velocity around a given axis. The vectorial change $\Delta\vec{v}$ in this vector during time Δt is (recall Figure 4.2 of Lecture 4)

$$\Delta\vec{v} = \Delta t\, \vec{\omega}\times\vec{v}$$

$$= \Delta t \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \omega^1 & \omega^2 & \omega^3 \\ v^1 & v^2 & v^3 \end{vmatrix}$$

Such a vectorial determinant can be generalized to higher dimensions. But, as far as I know, it will not represent a rotation in that case.

This is because the essential (= most consequential) property of the rotation process is a plane in which the rotation takes place, not around a unique normal. This plane is spanned by a bivector which arises as follows:

$$\begin{aligned}
\Delta\vec{v} &= \Delta t\, \vec{\omega}\times\vec{v} \\
&= \Delta t\left[e_1(\omega^2 v^3 - \omega^3 v^2) + e_2(\omega^3 v^1 - \omega^1 v^3) + e_3(\omega^1 v^2 - \omega^2 v^1)\right] \\
&= -\Delta t\left[\omega^1(e_2\otimes e_3 - e_3\otimes e_2) + \omega^2(e_3\otimes e_1 - e_1\otimes e_3) + \omega^3(e_1\otimes e_2 - e_2\otimes e_1)\right]\cdot\vec{v} \\
&= -\Delta t\left[\omega^1 e_2\wedge e_3 + \omega^2 e_3\wedge e_1 + \omega^3 e_3\wedge e_1\right]\cdot\vec{v} \qquad (19.1)
\end{aligned}$$

This change mathematizes an infinitesimal rotation. It is the sum of three rotations in each of planes spanned by the three pairs of basis vectors in the ambient Euclidean inner product space.

The bivectors $\{e_1\wedge e_2, e_2\wedge e_3, e_3\wedge e_1\}$ form a basis for a linear space. The ω's are the expansion coefficients for the linear combination, Eq. (19.1). It coordinate (i.e. observer) independence is becomes obvious when expressed as the trace of the product of the two antisymmetric matrices

$$\left[\mathcal{R}^{\ell m}\right] = \begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix} \text{ and } \left[e_m\wedge e_k\right] = \begin{bmatrix} 0 & -e_1\wedge e_2 & e_3\wedge e_1 \\ e_1\wedge e_2 & 0 & -e_2\wedge e_3 \\ -e_3\wedge e_1 & e_2\wedge e_3 & 0 \end{bmatrix}.$$

From the sum of the diagonal elements of their one verifies that Eq. (19.1) is the coordinate frame invariant

$$\Delta\vec{v} = \frac{\Delta t}{2!} R^{\ell m}\, \vec{e}_\ell \wedge \vec{e}_m \cdot \vec{v}$$ (Einstein Summation Convention)

$$\Delta\vec{v} = \Delta t\, R^{|\ell m|}\, \vec{e}_\ell \wedge \vec{e}_m \cdot \vec{v}$$ (Summation restricted to $\ell < m$)

By omitting reference to any particular vector $\vec{v}$ one arrives at the concept of rotation as a tensor of rank $\binom{2}{0}$.

Thus one has the following

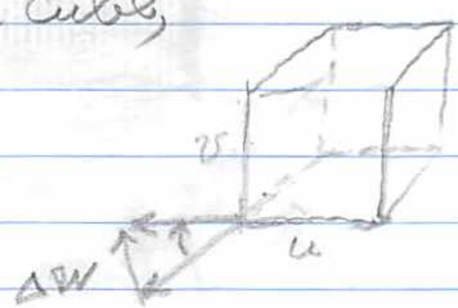
Definition ("rotation")

a) A rotation is a second rank antisymmetric tensor

$$\frac{\Delta t}{2!} R^{\ell m}\, e_\ell \wedge e_m = \Delta t\, \underbrace{e_\ell \wedge e_m\, R^{|\ell m|}}_{(\text{rotation}/\text{time})} = \text{"rotation"}$$

IV) Curvature as Rotation

The concept of rotation defined this way generalizes to four (and higher) dimensions of spaces with an inner product (i.e. metric structure). Indeed, applying it to the curvature-induced rotational change associated with the u-v spanned face of a cube,



one has

$$\begin{aligned}
\Delta \mathbf{W} &= \mathbf{e}_\ell\, w^k R^\ell{}_{k|\alpha\beta|}\, dx^\alpha \wedge dx^\beta (u,v) \\
&= \mathbf{e}_\ell\, w^k g_{km} R^{\ell m}{}_{|\alpha\beta|}\, dx^\alpha \wedge dx^\beta (u,v) \\
&= \mathbf{e}_\ell\, \underbrace{w^k \mathbf{e}_k}_{\mathbf{W}} \cdot \mathbf{e}_m\, \underset{\sim}{\mathcal{R}}^{\ell m}(u,v) \\
&= \mathbf{e}_\ell \otimes \mathbf{e}_m \cdot \mathbf{W}\, \underset{\sim}{\mathcal{R}}^{\ell m}(u,v)
\end{aligned}$$

Taking advantage of the curvature's metric-induced antisymmetry,

$R^{\ell m}{}_{\alpha\beta} = -R^{m\ell}{}_{\alpha\beta}$, on has

$$\Delta W = \tfrac{1}{2}(e_\ell \otimes e_m - e_m \otimes e_\ell) \cdot W R^{\ell m}(u,v)$$

$$= e_\ell \wedge e_m \cdot W R^{|\ell m|}(u,v)$$

Comparing this with the rotation defined on page 10-9, one arrives at

$$\boxed{e_\ell \wedge e_m R^{|\ell m|}(u,v) = \text{"rotation"}}$$

which is induced by the curvature in the area subtended by the vectors u and v.

This "rotation" is a $\binom{2}{0}$ tensor. For infinitesimal vectors u and v its components $R^{\ell m}(u,v)$ are the angles by which vector such as W get rotated in the plane spanned by e_l and e_m.

<u>Nota bene</u>: In the context of spacetime, the rotation can refer to Euclidean rotation, Lorentzian rotation or any of their combinations.