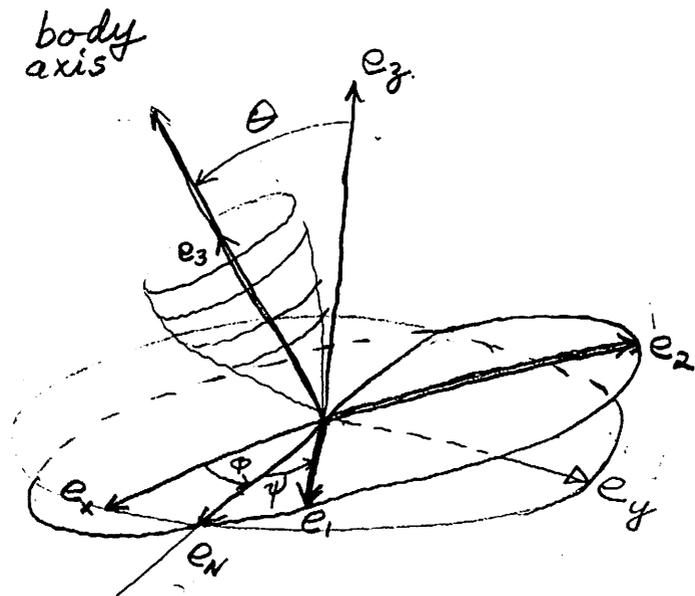


# Supplement to Lecture 24

Example: Solution to Problem 9.13  
in MTW

Rotation Group  $SO(3)$  as a  
manifold (Problem 9.13 in MTW) 19-1



Euler angles  $\phi, \theta$  &  $\psi$  of a rotating  
body.

$\{e_x, e_y, e_z\}$  orthonormal LAB basis

$\{e_1, e_2, e_3\}$  orthonormal BODY basis,  
principal axes of the body  
whose moments of inertia are,  
say,  $I_1, I_2,$  and  $I_3$

$e_N = e_2 \times e_3$  unit vector along the "line of nodes."

Problem 9.13 (Rotation Group: Its generators), direct attention to the following issues

9.13d) Question: Why does

$$P(\varphi, \theta, \psi) = R_{e_3}(\psi) R_{e_N}(\theta) R_{e_2}(\varphi)$$

define the Euler angles for a generic element  $P \in SO(3)$ ?

Answer:

Because: a)  $P(\varphi, \theta, \psi)$  is a product of three rotations which carries the orthonormal LAB frame basis into the orthonormal BODY frame basis:

$$\{e_x, e_y, e_z\} \xrightarrow{P} \{e_1, e_2, e_3\}$$

b) These three rotations are

(i) by an angle  $\varphi$  around  $e_3$  (i.e.  $e_3$  stays fixed) such that

$$R_{e_3}(\varphi) e_x \rightsquigarrow e_N$$

(ii) by an angle  $\theta$  around  $e_N$  such that

$$R_{e_N}(\theta) e_3 \rightsquigarrow e_2$$

(iii) by an angle  $\psi$  around  $e_3$  such that

$$R_{e_3}(\psi) e_N \rightsquigarrow e_1$$

and c) the domain of these angles is

$$0 < \varphi < 2\pi \quad 0 < \theta < \pi \quad 0 < \psi < 2\pi$$

The explicit form of rotations around the  $x$ ,  $y$ , and  $z$ -axes are as follows:

19-3b

$$R_x(\theta) = e^{K_1 \theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

$$R_y(\chi) = e^{K_2 \chi} = \begin{bmatrix} \cos \chi & 0 & -\sin \chi \\ 0 & 1 & 0 \\ \sin \chi & 0 & \cos \chi \end{bmatrix}$$

$$R_z(\phi) = e^{K_3 \phi} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where  $K_1$ ,  $K_2$  and  $K_3$  are antisymmetric matrices and  $0 \leq \theta, \chi, \phi \leq 2\pi$ .

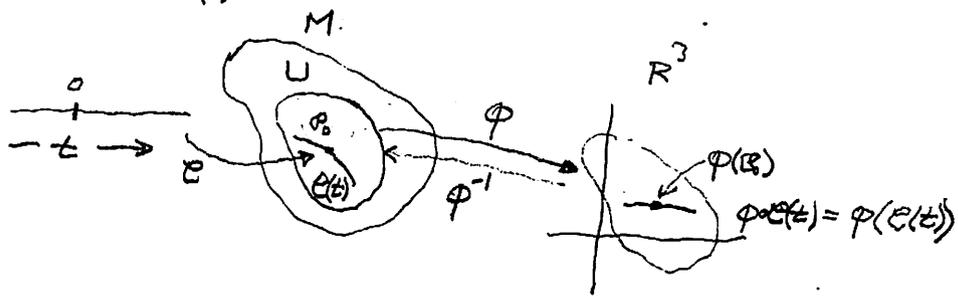
**Problem:** What is the explicit form of the rotation around the axis of an arbitrary unit vector with components  $(n_x, n_y, n_z)$ ?

These rotations are curves in  $M = SO(3)$ .

19-4

## The Manifold $SO(3)$ : The Construction of its Three Physically Induced Vector Fields.

The mathematical framework for grasping the geometrical and topological properties of  $SO(3)$ , the set of rotation matrices, is in terms a manifold and its coordinate chart(s).



Let us apply this constellation of concepts to Problem 9.13 in M T W.

19-5

There  $M = SO(3)$ , the set of rotation matrices is introduced via the three Euler angles, precession  $\phi$ , nutation  $\theta$ , and body spin angle  $\psi$ ;

$$P(\psi, \theta, \phi) = R_{e_2}(\psi) R_{e_1}(\theta) R_3(\phi) \in SO(3)$$

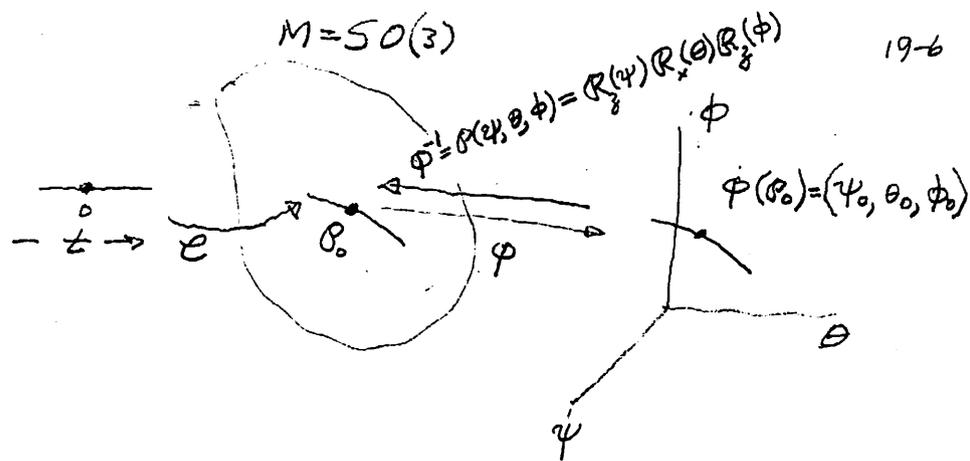
or in M T W's notation

$$P(\psi, \theta, \phi) = R_z(\psi) R_x(\theta) R_z(\phi) \in SO(3)$$

Comment: The "S" in  $SO(3)$  stands for "special", which means the rotation matrices under consideration have unit determinant,

$$\det P = 1.$$

The identification of a rotation in terms of these three Euler angles leads to the following formulation in terms of the manifold concept:



The rotations around the body axes  $\vec{e}_1$ ,  $\vec{e}_2$ , and  $\vec{e}_3$  by angle  $t$ , namely

$$R_{e_1}(t), R_{e_2}(t), \text{ and } R_{e_3}(t)$$

or using MTW notation

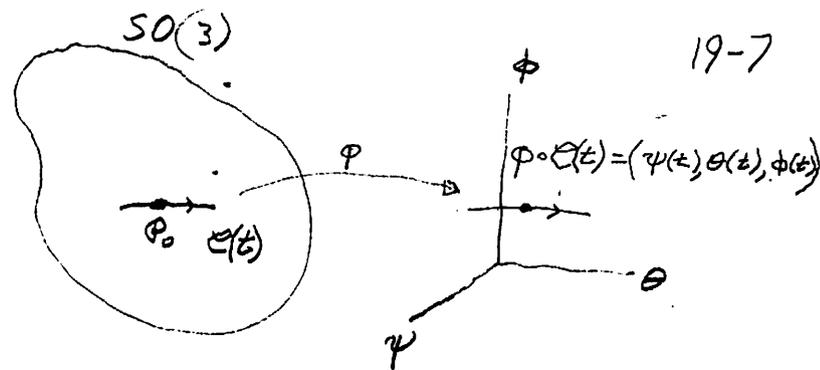
$$R_x(t), R_y(t), \text{ and } R_z(t)$$

lead to three different curves in  $SO(3)$

$$C_1(t) = R_{\vec{e}_1}(t) P_0 = R_x(t) P_0$$

$$C_2(t) = R_{\vec{e}_2}(t) P_0 = R_y(t) P_0$$

$$C_3(t) = R_{\vec{e}_3}(t) P_0 = R_z(t) P_0$$



The Euler coordinate representations of these curves are

$$\phi \circ C_1(t) = \phi \circ R_x(t) P_0 = (\psi_1(t), \theta_1(t), \phi_1(t))$$

$$\phi \circ C_2(t) = \phi \circ R_y(t) P_0 = (\psi_2(t), \theta_2(t), \phi_2(t))$$

$$\phi \circ C_3(t) = \phi \circ R_z(t) P_0 = (\psi_3(t), \theta_3(t), \phi_3(t))$$

Comments

1) For  $t = \Delta t \ll 1$  the curves  $C_i(t)$  are infinitesimal rotations around the body axes  $\vec{e}_i$   $i = 1, 2, 3$  respectively.

19-8

b) The Euler angle representations of each of these rotations expressed in terms of the principal linear part of its Taylor series expansion is therefore

$$(\psi_i(t), \theta_i(t), \phi_i(t)) = (\psi_0, \theta_0, \phi_0) + (\dot{\psi}_i(0), \dot{\theta}_i(0), \dot{\phi}_i(0))t + \left( \quad \right) \frac{t^2}{2!} + \dots$$

c) If  $t$  is a time parameter then Euler angular velocities  $\dot{\psi}$ ,  $\dot{\theta}$ , and  $\dot{\phi}$  are the spin angular frequency, the nutation angular freq'y, and precessional angular freq'y, respectively.

The above manifold formulation leads to the following problem (9.13 in MTW):

19-9

For each curve  $C_i(t)$  find the tangent vector at  $C_i(0) = P_0$ . Do this by

first finding the Euler angular velocities  $\dot{\psi}_i$ ,  $\dot{\theta}_i$ , and  $\dot{\phi}_i$  such that

$$C_i(\Delta t) \equiv R_{C_i}(\Delta t) P(\psi_0, \theta_0, \phi_0) = P(\psi_0 + \dot{\psi}_i \Delta t, \theta_0 + \dot{\theta}_i \Delta t, \phi_0 + \dot{\phi}_i \Delta t)$$

and

then showing that for  $f \in C^\infty(SO(3), P_0, R^1)$

$$\left. \frac{df(C_i(t))}{dt} \right|_{t=0} = \dot{\psi}_i(0) \frac{\partial f(P(\psi, \theta, \phi))}{\partial \psi} \Big|_{(\psi_0, \theta_0, \phi_0) = P_0} + \dot{\theta}_i(0) \frac{\partial f(P(\psi, \theta, \phi))}{\partial \theta} \Big|_{(\psi_0, \theta_0, \phi_0)} + \dot{\phi}_i(0) \frac{\partial f(P(\psi, \theta, \phi))}{\partial \phi} \Big|_{(\psi_0, \theta_0, \phi_0)}$$

$i = 1, 2, 3$

Comment:

Note that in light of

$$\varphi^{-1}(\psi, \theta, \phi) = P(\psi, \theta, \phi)$$

the  $\varphi$  coordinate representative of  $f$ ,

$$f_{\varphi \text{ rep}}(\psi, \theta, \phi) = f \circ \varphi^{-1}(\psi, \theta, \phi) = f(P(\psi, \theta, \phi)),$$

is a function defined on  $\mathbb{R}^3$  spanned by the three Euler angles  $0 \leq \psi, \theta, \phi \leq 2\pi$

19-10

Solution:

Assume we have found  $\psi, \theta, \phi$ . Then using the chain rule, we have

$$\begin{aligned} \frac{d}{dt} f(\psi(t), \theta(t), \phi(t)) &= \frac{d}{dt} f(\psi, \theta, \phi) \cdot \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} \\ &= \frac{d}{dt} f_{\text{free}}(\psi, \theta, \phi) + \frac{d}{dt} f_{\text{rep}}(\psi, \theta, \phi) \end{aligned}$$

$\left. \begin{array}{l} \text{def'n of } f_{\text{free}} \\ \text{def'n of } \psi, \theta, \phi \end{array} \right\} \text{ applied to } \psi, \theta, \phi$

$$\begin{aligned} &= \frac{d}{dt} f_{\text{free}}(\psi, \theta, \phi) + \frac{d}{dt} f_{\text{rep}}(\psi, \theta, \phi) \\ &= \frac{d}{dt} f_{\text{free}}(\psi, \theta, \phi) + \frac{d}{dt} f_{\text{rep}}(\psi, \theta, \phi) \end{aligned}$$

$\left. \begin{array}{l} \text{Taylor series on page 19-8} \\ \text{chain rule} \end{array} \right\}$

It follows that the three tangent vectors represented relative to the Euler coordinate basis  $\left\{ \frac{\partial}{\partial \psi}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \right\}$  are

$$\begin{aligned} \vec{e}_1(\theta) &\equiv \vec{e}_1(\psi, \theta, \phi) = \frac{\partial}{\partial \psi} + \phi \frac{\partial}{\partial \phi} \\ \vec{e}_2(\theta) &\equiv \vec{e}_2(\psi, \theta, \phi) = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial \phi} + \phi \frac{\partial}{\partial \phi} \\ \vec{e}_3(\theta) &\equiv \vec{e}_3(\psi, \theta, \phi) = \frac{\partial}{\partial \phi} \end{aligned}$$

Comment: The coefficients  $(\psi, \theta, \phi)$ ,  $i=1,2,3$  depend on  $(\psi, \theta, \phi)$ , the starting point of the curves  $\vec{e}_i(t)$ . There are several ways of determining these  $(\psi, \theta, \phi)$ -dependent coefficients. The result of this determination is (next page)

19-11

19-12

$$\begin{aligned}
 \mathbf{e}_1(\theta) &= -\sin\psi \cot\theta \frac{\partial}{\partial\psi} + \cos\psi \frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta} \frac{\partial}{\partial\phi} \\
 \mathbf{e}_2(\theta) &= -\cos\psi \cot\theta \frac{\partial}{\partial\psi} - \sin\psi \frac{\partial}{\partial\theta} + \frac{\cos\psi}{\sin\theta} \frac{\partial}{\partial\phi} \\
 \mathbf{e}_3(\theta) &= \frac{\partial}{\partial\psi}
 \end{aligned}$$

19-13

we shall integrate that vector field and determine the ("integral") curves associated with it.

GO TO LECTURE 20

### Summary

Rigid body rotations around a fixed point are expressible in terms of the three Euler angles  $\phi$ ,  $\theta$ , and  $\psi$ . These rotations are conceptualized as curves in the manifold  $SO(3)$ . The tangents to these curves are the vector fields  $\mathbf{e}_1(\theta)$ ,  $\mathbf{e}_2(\theta)$ , and  $\mathbf{e}_3(\theta)$  which are obtained mathematically by a process of differentiation as on page 19-10.

We shall now turn this process around and, starting with a given vector field,

Geometrical solution to

finding  $e_1(P)$ ,  $e_2(P)$ , and  $e_3(P)$ .

## ROTATION GROUP: GENERATORS

-1-

The 3 generators of rotations around the 3 principal axes of an arbitrarily oriented rigid body ("Rotation Group: Generators," Problem 9.13 in MTW)

Let  $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$  be a basis ("fixed basis") which is fixed relative to the fixed stars of  $\mathbb{R}^3$

Let  $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$  be a basis ("body basis") which is attached to the body.

These basis vectors may be thought of as being collinear with the three principal axes of the body.

The body's orientation is in general different from that of the fixed stars.

Let  $\vec{r}$  be a fixed point on the body, -2  
 Consequently, its representation is

$$\begin{aligned}\vec{r} &= x \vec{e}_x + y \vec{e}_y + z \vec{e}_z \\ &= x^1 \vec{e}_1 + x^2 \vec{e}_2 + x^3 \vec{e}_3\end{aligned}$$

relative to the lab (i.e. fixed star) and body basis respectively. Here  $x^1, x^2$  and  $x^3$  are constants because  $\vec{r}$  is fixed relative to the body.

The transformation

$$\rho: \{\vec{e}_x, \vec{e}_y, \vec{e}_z\} \xrightarrow{\rho} \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$$

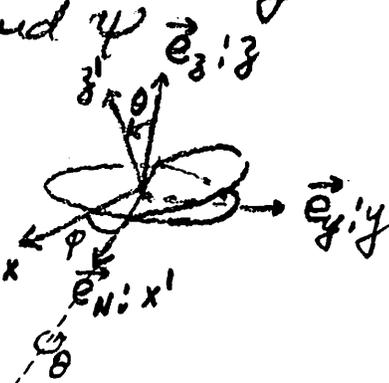
can be decomposed into three rotations by the three Euler angles  $\phi, \theta$  and  $\psi$

$$\rho(\phi, \theta, \psi) = R_{z'}(\psi) R_{x'}(\theta) R_z(\phi)$$

$$= R_z(\phi) R_x(\theta) R_{z'}(\psi)$$

In particular,

$$\vec{r}(\tau) = \rho(\phi, \theta, \psi) \vec{r}(0) = R_z(\phi) R_x(\theta) R_{z'}(\psi) \vec{r}(0).$$



Technical reminder: For an explicit computation it is best to work with the three rotations around the axes of a given orthonormal basis, such as the LAB basis, and not axes which are not...

such as  $\xi$  and  $\xi'$ .

-3-

○ We now consider three curves in  $SO(3)$ , each one passing through  $P(\varphi, \theta, \psi)$ .

$$R_{\vec{e}_1}(t) P(\varphi, \theta, \psi) \equiv C_1(t) \quad (\text{rotation around } \vec{e}_1)$$

$$R_{\vec{e}_2}(t) P(\varphi, \theta, \psi) \equiv C_2(t) \quad (\text{rotation around } \vec{e}_2)$$

$$R_{\vec{e}_3}(t) P(\varphi, \theta, \psi) \equiv C_3(t) \quad (\text{rotation around } \vec{e}_3)$$

Each of these rotations can be represented by three Euler angles. For small  $t$  one has

$$\begin{aligned} R_{\vec{e}_1}(t) P(\varphi, \theta, \psi) &= P(\varphi + at, \theta + bt, \psi + ct) \\ &= P(\varphi, \theta, \psi) + \left( a \frac{\partial P}{\partial \varphi} + b \frac{\partial P}{\partial \theta} + c \frac{\partial P}{\partial \psi} \right) t \end{aligned}$$

$$\begin{aligned} R_{\vec{e}_2}(t) P(\varphi, \theta, \psi) &= P(\varphi + dt, \theta + et, \psi + ft) \\ &= P(\varphi, \theta, \psi) + \left( d \frac{\partial P}{\partial \varphi} + e \frac{\partial P}{\partial \theta} + f \frac{\partial P}{\partial \psi} \right) t \end{aligned}$$

$$\begin{aligned} R_{\vec{e}_3}(t) P(\varphi, \theta, \psi) &= R_{\xi'}(t) R_{\xi}(\psi) R_{x'}(\theta) R_z(\varphi) \\ &= R_{\xi'}(\psi + t) R_{x'}(\theta) R_z(\varphi) \end{aligned}$$

question: What are the tangents to these curves at  $\mathcal{C}_1(0)$ ,  $\mathcal{C}_2(0)$ , and  $\mathcal{C}_3(0)$  respectively.

Answer:

$$\left. \frac{d\mathcal{C}_1(t)}{dt} \right|_{t=0} = \left( a \frac{\partial \rho}{\partial \varphi} + b \frac{\partial \rho}{\partial \theta} + c \frac{\partial \rho}{\partial \psi} \right)_{\rho(\varphi, \theta, \psi)}$$

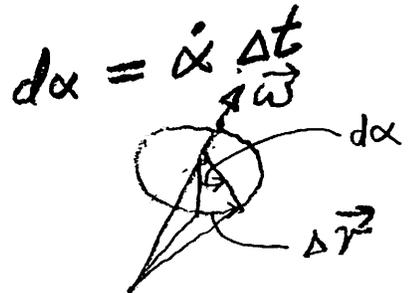
$$\left. \frac{d\mathcal{C}_2(t)}{dt} \right|_{t=0} = \left( d \frac{\partial \rho}{\partial \varphi} + e \frac{\partial \rho}{\partial \theta} + f \frac{\partial \rho}{\partial \psi} \right)_{\rho(\varphi, \theta, \psi)}$$

and by inspection

$$\left. \frac{d\mathcal{C}_3(t)}{dt} \right|_{t=0} = \left( 0 \frac{\partial \rho}{\partial \varphi} + 0 \frac{\partial \rho}{\partial \theta} + 1 \frac{\partial \rho}{\partial \psi} \right)_{\rho(\varphi, \theta, \psi)}$$

The remaining task consists of determining the coefficients  $a, b, \dots, f$ . This is achieved by taking advantage of the fact that infinitesimal rotations ( $t \equiv \Delta t \ll 1$ ) are representable as unit vectors and angles:

$$\begin{aligned} \Delta \vec{r} &= \vec{\omega} \times \vec{r} \Delta t \\ &= \vec{n}_{\omega} \times r \, d\alpha \quad \text{where} \\ &\quad \vec{n}_{\omega} \text{ unit vector} \end{aligned}$$



- 5 -

As a consequence, the effect of a generic infinitesimal rotation on a vector  $\vec{r}$  can be represented as a sum of three vectors, each one the respective effect of the infinitesimal rotation around the three axes of the three respective Euler angles:

$$\Delta \vec{r} \equiv \vec{n}_\omega \times \vec{r} d\alpha \quad d\alpha = \dot{\alpha} \Delta t$$
$$= \vec{e}_3 \times \vec{r} d\psi + \vec{e}_N \times \vec{r} d\theta + \vec{e}_3 \times \vec{r} d\phi$$

where  $d\psi = a \Delta t$

$$d\theta = b \Delta t$$

$$d\phi = c \Delta t$$

so that

$$\Delta \vec{r} = (a \vec{e}_3 + b \vec{e}_N + c \vec{e}_3) \times \vec{r} \Delta t$$

The application of the rotation matrix  $C_1(t=\Delta t)$  to a vector  $\vec{r}$  yields

$$C_1(\Delta t) \vec{r} = \begin{cases} R_{\vec{e}_1}(\Delta t) P(\varphi, \theta, \psi) \vec{r} & (*) \\ P(\varphi + d\varphi, \theta + d\theta, \psi + d\psi) \vec{r} & (**) \end{cases}$$

Eq. (\*) expresses a rotation of  $P(\varphi, \theta, \psi) \vec{r} \equiv P \vec{r}$  around the  $\vec{e}_1$ -axis by an angle  $\Delta t$ ! (\*)

$$C_1(\Delta t) \vec{r} = P \vec{r} + \vec{e}_1 \times \vec{r} \Delta t.$$

Eq. (\*\*) expresses three Euler angle rotations around  $\vec{e}_3, \vec{e}_N, \vec{e}_z$ !

$$C_1(\Delta t) \vec{r} = R_{\vec{e}_3}(\psi + a\Delta t) R_{\vec{e}_N}(\theta + b\Delta t) R_{\vec{e}_z}(\varphi + c\Delta t) \vec{r}$$

$$= P \vec{r} + \Delta t \left( a \frac{\partial P}{\partial \psi} + b \frac{\partial P}{\partial \theta} + c \frac{\partial P}{\partial \varphi} \right) \vec{r}$$

from the bottom of page 5

$$= P \vec{r} + \Delta \vec{r} = P \vec{r} + (a \vec{e}_3 + b \vec{e}_N + c \vec{e}_z) \times \vec{r} \Delta t \quad (**)$$

Eqs. (\*) and (\*\*) imply

$$\vec{e}_1 = a \vec{e}_z + b \vec{e}_N + c \vec{e}_3$$

(1)

Similarly, for the tangent to  $C_2(t)$  at  $t=0$

(see Page 4) one obtains

$$\vec{e}_2 = d\vec{e}_3 + e\vec{e}_N + f\vec{e}_3 \quad (2)$$

and finally, quite trivially

$$\vec{e}_3 = 0\vec{e}_3 + 0\vec{e}_N + 1\vec{e}_3 \quad (3)$$

Each boxed equation is 3 equations in the three unknowns  $(a, b, c)$  and  $(d, e, f)$  respectively. They are determined uniquely because  $\vec{e}_3, \vec{e}_N,$  and  $\vec{e}_3$  form a linearly independent set.

The three equations in the three unknowns are the components of the orthonormal triad

$$\{\vec{e}_3, \vec{e}_N, \vec{e}_3 \times \vec{e}_N = \vec{e}_1 \cos \psi + \vec{e}_2 \sin \psi\}$$

which are identified in the Figure,

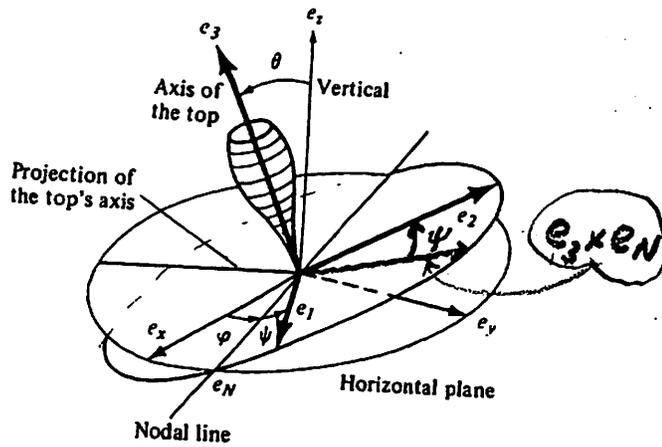


Figure 126 Euler angles

One makes the following observations

(i)  $\vec{e}_3$  lies in span  $\{\vec{e}_3, \vec{e}_3 \times \vec{e}_N\}$

$$\therefore \vec{e}_3 = e_3 \cos \theta + e_3 \times e_N \sin \theta$$

(ii)  $\vec{e}_1$  lies in span  $\{e_N, \vec{e}_3 \times \vec{e}_N\}$

$$\therefore \vec{e}_1 = e_N \cos \psi + e_3 \times e_N \sin \psi$$

(iii)  $\vec{e}_2$  lies in span  $\{e_N, \vec{e}_3 \times \vec{e}_N\}$

$$\therefore \vec{e}_2 = e_3 \times e_N \cos \psi - e_N \sin \psi$$

Now we are ready find the tangents  $\dot{e}_i(0)$  to  $e_i(t)$  at  $e_i(0)$  ( $i=1,2$ ) on  $P_4$ ,

① For  $\dot{e}_1(0)$  the boxed Eq. (1) on  $P_6$  becomes

$$e_N \cos \psi + e_3 \times e_N \sin \psi = a(e_3 \cos \theta + e_3 \times e_N \sin \theta) + b e_N + c e_3$$

Equating coefficients:

$$\left. \begin{array}{l} e_N: b = \cos \psi \\ e_3: c + a \cos \theta = 0 \\ e_3 \times e_N: \sin \psi = a \sin \theta \end{array} \right\} \Rightarrow \begin{array}{l} c = -\sin \psi \cot \theta \\ b = \cos \psi \\ a = \frac{\sin \psi}{\sin \theta} \end{array}$$

Thus the tangent on p 4

$$\left. \frac{d \mathcal{L}_1(t)}{dt} \right|_{t=0} = a \frac{\partial \mathcal{P}}{\partial \varphi} + b \frac{\partial \mathcal{P}}{\partial \theta} + c \frac{\partial \mathcal{P}}{\partial \psi}$$

is

Answer ①  $\left. \frac{d \mathcal{L}_1(t)}{dt} \right|_{t=0} = \frac{\sin \psi}{\sin \theta} \frac{\partial \mathcal{P}}{\partial \varphi} + \cos \psi \frac{\partial \mathcal{P}}{\partial \theta} - \sin \psi \cot \theta \frac{\partial \mathcal{P}}{\partial \psi} \equiv \mathcal{E}_1$

② Similarly, for  $\mathcal{L}_2(0)$  the boxed Eq. (2) on p 7 becomes

$$\vec{e}_3 \times \vec{e}_N \cos \psi - \vec{e}_N \sin \psi = d(\vec{e}_3 \cos \theta + \vec{e}_3 \times \vec{e}_N \sin \theta) + e \vec{e}_N + f \vec{e}_3$$

Equating the coefficients of the o.n. basis.

$$\left. \begin{array}{l} e_N: e = -\sin \psi \\ e_3: f + d \cos \theta = 0 \\ e_3 \times e_N: \cos \psi = d \sin \theta \end{array} \right\} \Rightarrow \begin{array}{l} f = -\cos \psi \cot \theta \\ e = -\sin \psi \\ d = \frac{\cos \psi}{\sin \theta} \end{array}$$

Thus

$$\dot{\mathcal{L}}_2(t) \Big|_{t=0} = d \frac{\partial \mathcal{P}}{\partial \varphi} + e \frac{\partial \mathcal{P}}{\partial \theta} + f \frac{\partial \mathcal{P}}{\partial \psi}$$

Answer ②  $\left. \dot{\mathcal{L}}_2(t) \right|_{t=0} = \frac{\cos \psi}{\sin \theta} \frac{\partial \mathcal{P}}{\partial \varphi} - \sin \psi \frac{\partial \mathcal{P}}{\partial \theta} - \cos \psi \cot \theta \frac{\partial \mathcal{P}}{\partial \psi} \equiv \mathcal{E}_2$

③ Finally, Differs by a minus sign from MTW p 243

Answer ③  $\left. \mathcal{L}_3(t) \right|_{t=0} = \frac{\partial \mathcal{P}}{\partial \psi} \equiv \mathcal{E}_3(\mathcal{P})$

## Euler's Theorem:

The general displacement of a rigid body with one point fixed is a rotation about some axis. Furthermore, this rotation is the product of three rotations by three unique Euler angles.

Proof: 1.) The orientation of a body is determined by three basis vectors.

2.) Being rigid, the body has its three o.n. basis vectors reoriented by that displacement into another set of three o.n. basis vectors.

We know from linear algebra that such a transformation takes an o.n. basis into an o.n. basis  $\Leftrightarrow$  the transformation is a rotation, i.e. it is expressed by an orthogonal matrix

$$R: R^T = R^{-1} \quad \text{has eigenvalues } |\lambda| = 1,$$

3.) In three dimensional vector space  $R$  has one eigenvector whose eigenvalue  $\lambda = 1$ :

$$R \vec{n} = \vec{n}$$

i.e.  $\det(R - I) = 0$  always whenever  $\dim(\text{vector space}) = \text{odd}$

$$\begin{array}{ccc} \begin{matrix} \text{o.n.} \\ \{e_i\} \end{matrix} & \xrightarrow{R} & \begin{matrix} \text{o.n.} \\ \{e'_i\} \end{matrix} \\ \text{o.n.} & & \text{o.n.} \\ & \Downarrow & \\ & R^T = R^{-1} & \end{array}$$

Consequently,

$$P = R_{z'}(\psi) R_{x'}(\theta) R_z(\phi)$$

$$P = R_{z'}(\psi) R_z(\phi) R_{x'}(\theta) \underbrace{R_z(-\phi) R_z(\phi)}_{I} \quad (*)$$

$I = \text{identity}$

Similarly  $R_{z'}(\psi)$  is a rotation which is related to  $R_z(\psi)$  by the coordinate transformation

$R_z(\phi) R_{x'}(\theta)$  which relates the component of  $b$  in the  $z$ -frame to those in the  $z'$ -frame!

$$R_z(\phi) R_{x'}(\theta) b \quad \text{components of } \vec{b} \text{ relative to } \text{span}\{e_z, e_x, e_z \times e_x\}$$

In that frame  $R_z(\psi)$  yields the component of  $R_z(\psi) b$  relative to the  $z'$  frame. Thus one obtains

$$R_{z'}(\psi) b = [R_z(\phi) R_{x'}(\theta)]^{-1} R_z(\psi) R_{x'}(\theta) R_z(\phi) b$$

$$\text{or } R_{z'}(\psi) = R_z(\phi) R_{x'}(\theta) R_z(\psi) [R_z(\phi) R_{x'}(\theta)]^{-1}$$

Inserting this into the boxed Eq. (\*), one has

$$P(\phi, \theta, \psi) = R_z(\phi) R_{x'}(\theta) R_{z'}(\psi)$$

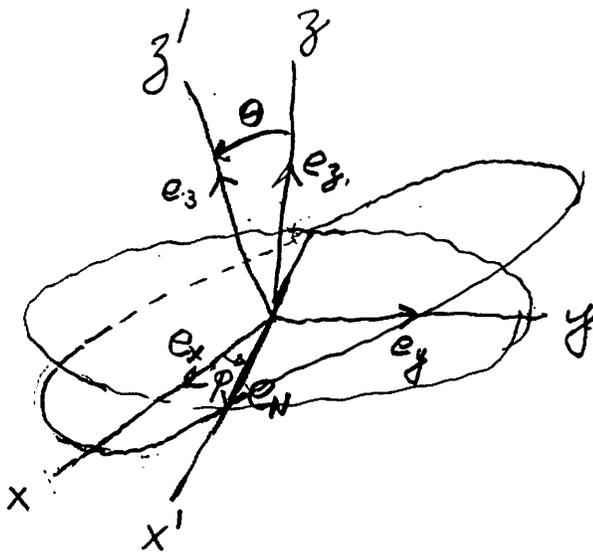
Thus, dropping the "primes" demands that the three corresponding rotations are to be taken in reverse order.

4.) Let  $R$  be that displacement of a rigid body (with one point fixed) which takes the LAB basis into BODY basis

$$\{e_x, e_y, e_z\} \xrightarrow{R} \{e_1, e_2, e_3\}$$

As shown in the Figure on Page 19-1, this rotation is the product of three rotations by three respective Euler angles

$$R = R_{z'}(\psi) R_{x'}(\theta) R_z(\phi) \equiv P(\psi, \theta, \phi).$$



These three rotations are around the  $z$ -axis,  $x'$ -axis and  $z'$ -axis by  $\phi$ ,  $\theta$ , and  $\psi$  respectively.

Discussion:

The problem with the matrix  $P$  is that  $R_{z'}$  and  $R_{x'}$  represent rotations around the  $z'$  and  $x'$  (i.e. BODY) axes. But we need to express everything in terms of rotations around the  $x$  and  $z$  axes of the LAB.

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \end{bmatrix}$$

rotation around x-axis expressed relative to LAB basis.

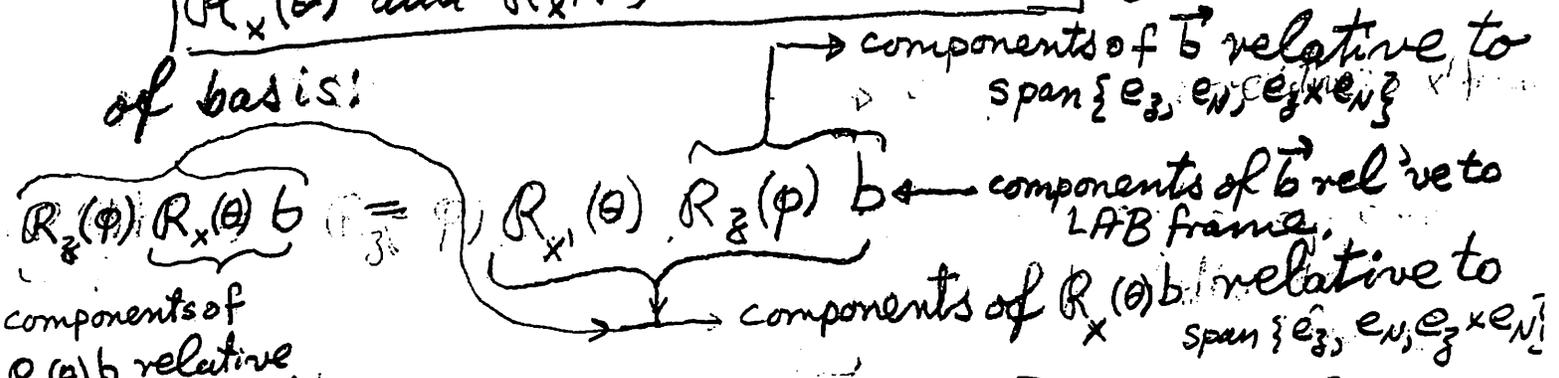
and

$$R_z(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To achieve this we observe that

$R_x(\theta)$  and  $R_{x'}(\theta)$  are related by a change

of basis:



components of  $R_x(\theta) \vec{b}$  relative to LAB basis

This holds for all vectors  $\vec{b}$ . It follows that

$$R_z(\phi) R_x(\theta) R_z(-\phi) = R_{x'}(\theta)$$

Concluding comment:

The rotation axes of the product rotation

$$P = R_z(\psi) R_x(\theta) R_z(\phi)$$

refers to the BODY axes, not to the LAB axes, as in our development