Example: Solution to Problem 9.13 in MTW

Rotation Group \( SO(3) \) as a manifold (Problem 9.13 in MTW)

Euler angles \( \phi, \theta, \psi \) of a rotating body

\[ \{e_1, e_2, e_3\} \text{ orthonormal LAB basis} \]

\[ \{e_1', e_2', e_3'\} \text{ orthonormal BODY basis} \]

principal axes of the body whose moments of inertia are say \( I_1, I_2, \) and \( I_3 \)
Problem 9.13 (Rotation Group $S_3$ generated)

Question: Why does $\mathbb{R}^3 = \mathbb{R}^e(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ have direct attention to the following issues?

(i) Every unit vector along the line of nodes.

$\mathbf{e}_i \in \mathbb{R}^3$ unit vector along the line of nodes.

(ii) By an angle $\theta$ around $\mathbf{e}_3$, such that $0 < \theta < 2\pi$.

(iii) By an angle $\phi$ around $\mathbf{e}_1$, such that $0 < \phi < 2\pi$.

(iv) By an angle $\psi$ around $\mathbf{e}_2$, such that $0 < \psi < 2\pi$.

(v) Define the Euler angle for a generic element $\mathbf{R}(\theta, \phi, \psi)$.

Because $\mathbb{R}^3 = \mathbb{R}^e(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a product of three rotations, which carried the orthornormal LAB frame basis and the orthonormal body frame basis.

(iii) $\mathbf{e}_3 \rightarrow \mathbf{e}_3$

(b) These three rotations are $(\theta, \phi, \psi) \rightarrow \mathbf{e}_i$, $\mathbf{e}_j$, $\mathbf{e}_k$.
The explicit form of rotations around the $x, y,$ and $z$-axes are as follows:

\[ R_x(\theta) = e^{K_1 \theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \]

\[ R_y(\chi) = e^{K_2 \chi} = \begin{bmatrix} \cos \chi & 0 & -\sin \chi \\ 0 & 1 & 0 \\ \sin \chi & 0 & \cos \chi \end{bmatrix} \]

\[ R_z(\phi) = e^{K_3 \phi} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

where $K_1, K_2$ and $K_3$ are antisymmetric matrices and $0 \leq \theta, \chi, \phi \leq 2\pi$.

Problem: What is the explicit form of the rotation around the axis of an arbitrary unit vector with components $(\eta_x, \eta_y, \eta_z)$? These rotations are curves in $M = SO(3)$. 
The Manifold $SO(3)$: The Construction of its Three Physically Induced Vector Fields.

The mathematical framework for grasping the geometrical and topological properties of $SO(3)$, the set of rotation matrices, is in terms of a manifold and its coordinate chart(s).

Let us apply this constellation of concepts to Problem 9.13 in MTW.

There $M = SO(3)$, the set of rotation matrices, is introduced via the three Euler angles, precession $p$, nutation $q$, and body spin angle $\psi$;

\[ \mathbf{R}(\psi, \theta, \phi) = \left( \begin{array}{ccc} R_1(\psi) & R_2(\theta) & R_3(\phi) \end{array} \right) \in SO(3) \]

or in MTW's notation

\[ \mathbf{R}(\psi, \theta, \phi) = \left( \begin{array}{ccc} R_1(\psi) & R_2(\theta) & R_3(\phi) \end{array} \right) \in SO(3) \]

Comment: The "$S" in SO(3) stands for "special", which means the rotation matrices under consideration have unit determinant

\[ \det \mathbf{R} = 1 \]

The identification of a rotation in terms of the three Euler angles leads to the following formulation in terms of the manifold concept.
The rotations around the body axes $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ by angle $\tau$, namely

$R_{\vec{e}_1}(t)$, $R_{\vec{e}_2}(t)$, and $R_{\vec{e}_3}(t)$

or using MTW notation

$R_x(t)$, $R_y(t)$, and $R_z(t)$

lead to three different curves in $SO(3)$

$C_1(t) = R_{\vec{e}_1}(t)P_0 = R_{\vec{e}_1}(t)P_0$

$C_2(t) = R_{\vec{e}_2}(t)P_0 = R_{\vec{e}_2}(t)P_0$

$C_3(t) = R_{\vec{e}_3}(t)P_0 = R_{\vec{e}_3}(t)P_0$

The Euler coordinate representations of these curves are

$\Phi \cdot C_1(t) = \Phi \cdot R_{\vec{e}_1}(t)P_0 = (\psi_1(t), \Theta_1(t), \Phi_1(t))$

$\Phi \cdot C_2(t) = \Phi \cdot R_{\vec{e}_2}(t)P_0 = (\psi_2(t), \Theta_2(t), \Phi_2(t))$

$\Phi \cdot C_3(t) = \Phi \cdot R_{\vec{e}_3}(t)P_0 = (\psi_3(t), \Theta_3(t), \Phi_3(t))$

Comments:

9) For $t = \Delta t \ll 1$ the curves $C_i(t)$ are infinitesimal rotations around the body axes $\vec{e}_i$, $i = 1, 2, 3$ respectively.
b) The Euler angle representations of each of these rotations expressed in terms of the principal linear part of its Taylor series expansion is therefore:

\[
(\psi(t), \theta(t), \phi(t)) = (\psi_0, \theta_0, \phi_0) + (\dot{\psi}_0 t, \dot{\theta}_0 t, \dot{\phi}_0 t) t + \cdots
\]

\[
+ (\cdots) \frac{t^2}{2} + \cdots
\]

c) If \( t \) is a time parameter then Euler angular velocities \( \dot{\psi}, \dot{\theta}, \text{ and } \dot{\phi} \) are the spin angular frequency, the nutation angular frequency, and precessional angular frequency, respectively.

The above manifold formulation leads to the following problem (8.13 in MTW):

For each curve \( C(\mathbb{R}) \) find the tangent vector at \( C(0) = (0) \). Do this by first finding the Euler angular velocities \( \dot{\psi}, \dot{\theta}, \text{ and } \dot{\phi} \) such that

\[
C(\mathbb{R}) = \mathcal{R}(\mathbb{R}) \mathcal{P}(\psi, \theta, \phi) = \mathcal{P}(\psi + \dot{\psi} \Delta t, \theta + \dot{\theta} \Delta t, \phi + \dot{\phi} \Delta t)
\]

and then showing that for \( f \in C(\mathbb{R}, \mathbb{R}) \)

\[
\frac{df(C(\mathbb{R}))}{dt} \bigg|_{t=0} = \psi(t) \frac{\partial f}{\partial \psi} + \theta(t) \frac{\partial f}{\partial \theta} + \phi(t) \frac{\partial f}{\partial \phi}
\]

Comment:
Note that in light of

\( \Phi'(\psi, \theta, \phi) = \mathcal{P}(\psi, \theta, \phi) \cdot \Phi(\psi, \theta, \phi) \)

the \( \phi \) coordinate representation of \( f \)

\( f_{\Phi \phi}(\psi, \theta, \phi) = f \cdot \Phi'(\psi, \theta, \phi) = f(\mathcal{P}(\psi, \theta, \phi)) \)
The result of this determination
(φ, θ) - dependent coefficients.
We use of determining shears.
There are several curves C(t). The starting point of the
on (φ, θ) point depends
The coefficients (φ, θ) depend
\[ \begin{align*}
\phi_0 & = \frac{\phi}{2} + \frac{\theta}{2} + \frac{\phi_0}{2} + \frac{\theta_0}{2} \\
\phi_1 & = \phi + \phi_0 \\
\phi_2 & = \phi_0 + \phi_0 \\
\end{align*} \]

\[ \text{Euler coordinate basis} \quad \left\{ \begin{array}{c}
\phi \\
\theta \\
\end{array} \right\} \]

\[ \text{Assume you have found } \phi, \theta \]

Then using the theorem, we have

It follows that the three tangent

Assume you have found \( \phi, \theta \). Then using the

Assume you have found \( \phi, \theta \).

The three Euler angles, namely \( \phi, \theta, \phi \), are

\[ \phi, \theta, \phi \]
\[ E_1(\phi) = -\sin \psi \cot \theta \frac{\partial}{\partial \psi} + \cos \psi \frac{\partial}{\partial \theta} + \sin \phi \frac{\partial}{\partial \phi} \]
\[ E_2(\phi) = \cos \psi \cot \theta \frac{\partial}{\partial \psi} - \sin \psi \frac{\partial}{\partial \theta} + \cos \phi \frac{\partial}{\partial \phi} \]
\[ E_3(\phi) = \frac{\partial}{\partial \psi} \]

we shall integrate that vector field and determine the ("integral") curves associated with it.

GO TO LECTURE 20

Summary:
Rigid body rotations around a fixed point are expressible in terms of the three Euler angles \(\phi, \theta,\) and \(\psi\). These rotations are conceptualized as curves in the manifold \(SO(3)\). The tangents to these curves are the vector fields \(E_1(\phi), E_2(\phi),\) and \(E_3(\phi)\) which are obtained mathematically by a process of differentiation as on page 19-10.

We shall now turn this process around and, starting with a given vector field,
Geometrical solution to finding $E_1(P)$, $E_2(P)$, and $E_3(P)$. 
The 3 generators of rotations around the 3 principal axes of an arbitrarily oriented rigid body ("Rotation Group: Generators" Problem 9.13 in MTW)

Let \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} be a basis ("fixed basis") which is fixed relative to the fixed stars of \(\mathbb{R}^3\).

Let \{\vec{e}_1', \vec{e}_2', \vec{e}_3'\} be a basis ("body basis") which is attached to the body.

These basis vectors may be thought of as being collinear with the three principal axes of the body.

The body's orientation is in general different from that of the fixed stars.
Let \( \vec{r} \) be a fixed point on the body. Consequently, its representation is
\[
\vec{r} = x \vec{e}_x + y \vec{e}_y + z \vec{e}_z
\]
\[
= x' \vec{e}_1 + x^2 \vec{e}_2 + x^3 \vec{e}_3
\]
relative to the lab (i.e., fixed star) and body basis respectively. Here \( x', x^2 \) and \( x^3 \) are constants because \( \vec{r} \) is fixed relative to the body.

The transformation
\[
\mathbf{R} : \{ \vec{e}_x, \vec{e}_y, \vec{e}_z \} \rightarrow \{ \vec{\bar{e}}_1, \vec{\bar{e}}_2, \vec{\bar{e}}_3 \}
\]
can be decomposed into three rotations by the three Euler angles \( \phi, \theta \) and \( \psi \):
\[
\mathbf{R}(\phi, \theta, \psi) = R_{31}(\psi) R_{x1}(\theta) R_{z1}(\phi)
\]
\[
= R_{z1}(\phi) R_{x1}(\theta) R_{z1}(\psi) \vec{e}_x \rightarrow \vec{\bar{e}}_3
\]

In particular,
\[
\vec{r}(\tau) = \mathbf{R}(\phi, \theta, \psi) \vec{r}(0) = R_{z1}(\phi) R_{x1}(\theta) R_{z1}(\psi) \vec{r}(0).
\]

Technical reminder: For an explicit computation it is best to work with the three rotations around the axes of a given orthogonal normal basis, such as the LAB basis, and not axes which are rotated.
such as $\alpha$ and $\beta$.

We now consider three curves in $SO(3)$, each one passing through $P(\varphi, \theta, \psi)$.

$R_{e_1}(t) P(\varphi, \theta, \psi) = C_1(t)$ (rotation around $e_1$)

$R_{e_2}(t) P(\varphi, \theta, \psi) = C_2(t)$ (rotation around $e_2$)

$R_{e_3}(t) P(\varphi, \theta, \psi) = C_3(t)$ (rotation around $e_3$)

Each of these rotations can be represented by three Euler angles. For small $t$ one has

$R_{e_1}(t) P(\varphi, \theta, \psi) = P(\varphi + at, \theta + bt, \varphi + ct)$

$= P(\varphi, \theta, \psi) + \left( \frac{a}{\varphi} \frac{\partial P}{\partial \varphi} + \frac{b}{\varphi} \frac{\partial P}{\partial \theta} + \frac{c}{\varphi} \frac{\partial P}{\partial \psi} \right) t$

$R_{e_2}(t) P(\varphi, \theta, \psi) = P(\varphi + dt, \theta + et, \varphi + ft)$

$= P(\varphi, \theta, \psi) + \left( \frac{d}{\varphi} \frac{\partial P}{\partial \varphi} + \frac{e}{\varphi} \frac{\partial P}{\partial \theta} + \frac{f}{\varphi} \frac{\partial P}{\partial \psi} \right) t$

$R_{e_3}(t) P(\varphi, \theta, \psi) = R_3(y(t)) R_3(\psi(t)) R_x(\theta(t)) R_3(\varphi)$

$= R_3(\psi + t) R_x(\theta) R_3(\varphi)$
The question is: What are the tangents to these curves at $C_1(0)$, $C_2(0)$, and $C_3(0)$ respectively.

Answer:

$$\left. \frac{dC_1(t)}{dt} \right|_{t=0} = \left( a \frac{\partial E}{\partial \phi} + b \frac{\partial E}{\partial \theta} + c \frac{\partial E}{\partial \psi} \right)_{\phi_0, \theta_0, \psi_0}$$

$$\left. \frac{dC_2(t)}{dt} \right|_{t=0} = \left( d \frac{\partial E}{\partial \phi} + e \frac{\partial E}{\partial \theta} + f \frac{\partial E}{\partial \psi} \right)_{\phi_0, \theta_0, \psi_0}$$

and by inspection

$$\left. \frac{dC_3(t)}{dt} \right|_{t=0} = \left( 0 \frac{\partial E}{\partial \phi} + 0 \frac{\partial E}{\partial \theta} + 1 \frac{\partial E}{\partial \psi} \right)_{\phi_0, \theta_0, \psi_0}$$

The remaining task consists of determining the coefficients $a, b, \ldots, f$. This is achieved by taking advantage of the fact that infinitesimal rotations ($t = \Delta t << 1$) are representable as unit vectors and angles:

$$\Delta \vec{r} = \vec{\omega} \times \vec{r} \Delta t$$

$$= \vec{n} \times r \, d\alpha$$

where $d\alpha = \vec{\alpha} \cdot \Delta \vec{t}$.

The unit vector
As a consequence, the effect of a generic infinitesimal rotation on a vector \( \vec{r} \) can be represented as a sum of three vectors, each one the respective effect of the infinitesimal rotation around the three axes of the three respective Euler angles:

\[
\Delta \vec{r} = \vec{r} \times \vec{x} \, d\alpha : \quad d\alpha = \alpha \, dt \\
= \vec{e}_3 \times \vec{r} \, d\psi + \vec{e}_n \times \vec{r} \, d\theta + \vec{e}_3 \times \vec{r} \, d\phi
\]

where \( d\psi = a \, dt \)
\( d\theta = b \, dt \)
\( d\phi = c \, dt \)

so that

\[
\Delta \vec{r} = \left( a \vec{e}_3 + b \vec{e}_n + c \vec{e}_3 \right) \times \vec{r} \, dt
\]
The application of the rotation matrix \( E_1(t=\Delta t) \) to a vector \( \vec{r} \) yields

\[
E_1(\Delta t) \vec{r} = \begin{cases} 
R_{\vec{e}_1}(\Delta t) \vec{P}(\phi, \theta, \psi) \vec{r} & (\star) \\
\vec{P}(\phi+\Delta \phi, \theta+\Delta \theta, \psi+\Delta \psi) \vec{r} & (\star\star) 
\end{cases}
\]

Eq. (\( \star \)) expresses a rotation of \( \vec{P}(\phi, \theta, \psi) \vec{r} \equiv \vec{P} \vec{r} \) around the \( \vec{e}_1 \)-axis by an angle \( \Delta t \):

\[
E_1(\Delta t) \vec{r} = \vec{P} \vec{r} + \vec{e}_1 \times \vec{r} \Delta t. 
\]

Eq. (\( \star\star \)) expresses three Euler angle rotations around \( \vec{e}_3, \vec{e}_N, \vec{e}_3 \):

\[
E_1(\Delta t) \vec{r} = R_{\vec{e}_3}(\psi+\Delta \psi) R_{\vec{e}_N}(\theta+\Delta \theta) R_{\vec{e}_3}(\phi+\Delta \phi) \vec{r}
\]

\[
= \vec{P} \vec{r} + \Delta \vec{r}
\]

\[
= \vec{P} \vec{r} + (a \vec{e}_3 + b \vec{e}_N + c \vec{e}_3) \times \vec{r} \Delta t 
\]

Eqs. (\( \star \)) and (\( \star\star \)) imply

\[
\vec{e}_1 = a \vec{e}_3 + b \vec{e}_N + c \vec{e}_3 
\]

(1)
Similarly, for the tangent to $C_2(t)$ at $t = 0$ (see page 4) one obtains

$$\bar{e}_2 = d\bar{e}_3 + e\bar{e}_n + f\bar{e}_3$$

(2)

and finally, quite trivially

$$\bar{e}_3 = 0\bar{e}_3 + 0\bar{e}_n + 1\bar{e}_3$$

(3)

Each boxed equation is 3 equations in the three unknowns $(a, b, c)$ and $(d, e, f)$ respectively. They are determined uniquely because $\bar{e}_3$, $\bar{e}_n$, and $\bar{e}_3$ form a linearly independent set.

The three equations in the three unknowns are the components of the orthonormal triad

$$\{\bar{e}_3, \bar{e}_n, \bar{e}_3 \times \bar{e}_n = \bar{e}_1 \cos \psi + \bar{e}_2 \sin \psi\}$$

which are identified in the figure.
One makes the following observations

(i) \( \tilde{e}_3 \) lies in span \( \{ \tilde{e}_3, \tilde{e}_3 \times \tilde{e}_N \} \)

\[ \tilde{e}_3 = e_3 \cos \theta + e_3 \times e_N \sin \theta. \]

(ii) \( \tilde{e}_1 \) lies in span \( \{ e_N, \tilde{e}_3 \times \tilde{e}_N \} \)

\[ \tilde{e}_1 = e_N \cos \varphi + e_3 \times e_N \sin \varphi. \]

(iii) \( \tilde{e}_2 \) lies in span \( \{ e_N, \tilde{e}_3 \times \tilde{e}_N \} \)

\[ \tilde{e}_2 = e_3 \times e_N \cos \varphi - e_N \sin \varphi. \]

Now we are ready find the tangents \( \tilde{e}_i(t) \) to \( \tilde{e}_i(t) \)
at \( \tilde{e}_i(0) \) \((i=1,2)\) on P 4.

1. For \( \tilde{e}_1(0) \) the boxed Eq. (1) on P 6 becomes

\[
e_N \cos \varphi + e_3 \times e_N \sin \varphi = a(e_3 \cos \theta + e_3 \times e_N \sin \theta) + b e_N + c e_3
\]

Equating coefficients:

\[
e_N: \quad b = \cos \varphi
\]

\[
e_3: \quad c + a \cos \theta = 0 \]

\[
e_3 \times e: \quad \sin \varphi = a \sin \theta \]

\[ \Rightarrow \ \left\{ \begin{array}{l} c = -\sin \varphi \cot \theta \\ b = \cos \varphi \\
 a = \frac{\sin \varphi}{\sin \theta} \end{array} \right. \]
Thus the tangent on p 4
\[ \frac{d C_1(t)}{dt} \bigg|_{t=0} = a \frac{\partial \Phi}{\partial \varphi} + b \frac{\partial \Phi}{\partial \theta} + c \frac{\partial \Phi}{\partial \psi} \]

is
\[ \frac{d C_1(t)}{dt} \bigg|_{t=0} = \frac{\sin \psi}{\sin \theta} \frac{\partial \Phi}{\partial \varphi} + \cos \psi \frac{\partial \Phi}{\partial \theta} - \sin \psi \cot \theta \frac{\partial \Phi}{\partial \psi} \]

Answer 0

(2) Similarly, for \( C_2(0) \) the boxed Eq. (2) on p 7 becomes
\[ e_3 \times e_N \cos \psi - e_N \sin \psi = d (e_3 \cos \theta + e_3 \times e_N \sin \theta) + e e_N + f \bar{e}_3 \]

Equating the coefficients of the o.n. basis,

\[ e_N: \quad e = -\sin \psi \quad f = -\cos \psi \cot \theta \]
\[ e_3: \quad f + d \sin \theta = 0 \quad \implies \quad e = -\sin \psi \]
\[ e_3 \times e_N: \quad \cos \psi = d \sin \theta \quad d = \frac{\cos \psi}{\sin \theta} \]

Thus
\[ C_2(t) \bigg|_{t=0} = d \frac{\partial \Phi}{\partial \varphi} + e \frac{\partial \Phi}{\partial \theta} + f \frac{\partial \Phi}{\partial \psi} \]

Answer 0

\[ C_2(t) \bigg|_{t=0} = \frac{\cos \psi}{\sin \theta} \frac{\partial \Phi}{\partial \varphi} - \sin \psi \frac{\partial \Phi}{\partial \theta} - \cos \psi \cot \theta \frac{\partial \Phi}{\partial \psi} = e_2(\theta) \]

(3) Finally, differs by a minus sign from MTW p 243

Answer 3

\[ C_3(t) \bigg|_{t=0} = \frac{\partial \Phi}{\partial \psi} = e_3(\theta) \]
Supplement to \text{SO}(3).

Euler's Theorem:

The general displacement of a rigid body with one point fixed is a rotation about some axis. Furthermore this rotation is the product of three rotation by three unique Euler angles.

Proof: 1) The orientation of a body is determined by three basis vectors.
2) Being rigid, the body has its three o.n. basis vectors reoriented by the displacement into another set of three o.n. basis vectors.

We know from linear algebra that a transformation takes an o.n. i basis into an o.n. basis \iff the transformation is a rotation, i.e. it is expressed by an orthogonal matrix \( R \): \( R^T = R^{-1} \) has eigenvalues \( |\lambda| = 1 \).

3) In three dimensional vector space \( R \) has one eigenvector whose eigenvalue \( \lambda = 1 \):

\( R \overrightarrow{n} = \overrightarrow{n} \)

i.e. \( \det[R-I] = 0 \) always whenever \( \dim(\text{vector space}) = 3 \).
Consequently,

\[ P = R_z(\psi) R_x(\theta) R_z(\phi) \]

\[ \mathbf{R} = R_z(\psi) R_z(\phi) R_x(\theta) R_z(-\phi) R_z(\psi) \]  \hspace{1cm} (\star)

I = identity

Similarly, \( R_z(\psi) \) is a rotation which is related to \( R_z(\psi) \) by the coordinate transformation \( R_z(\psi) R_x(\theta) \) which relates the component of \( b \)
in the \( z' \)-frame to those in the \( z \)-frame.

\[ R_z(\phi) R_x(\theta) b, \ span \{e_3, e_4, e_5 \times e_4 \} \]

In that frame \( R_z(\psi) \) yields the component of \( R_z(\psi) b \) relative to the \( z' \)-frame. Thus one obtains

\[ R_z(\psi) b = \left[ R_z(\Psi) R_x(\theta) \right] R_z(\psi) R_z(\phi) R_x(\theta) b \]

or

\[ R_z(\psi) = R_z(\psi) R_x(\theta) R_z(\phi) \left[ R_z(\phi) R_x(\theta) \right]^{-1} \]

Inserting this into the boxed Eq. (\star), one has

\[ P(\phi, \theta, \psi) = R_z(\phi) R_x(\theta) R_z(\psi) \]

Thus, dropping the "primes" demands that the three corresponding rotations be to be taken in reverse order.
4.) Let \( R \) be that displacement of a rigid body (with one point fixed) which takes the LAB basis into BODY basis

\[ \{e_x, e_y, e_z\} \xrightarrow{R} \{e_1, e_2, e_3\} \]

As shown in the Figure on Page 19-1, this rotation is the product of three rotations by three respective Euler angles

\[ R = R_z(\psi) R_y(\theta) R_z(\phi) \equiv R(\psi, \theta, \phi) . \]

These three rotations are around the \( z \)-axis, \( x' \)-axis and \( z' \)-axis by \( \psi, \theta, \) and \( \phi \) respectively.
Discussion:
The problem with the matrix $\mathbf{R}$ is that $R_y$ and $R_x$, represent rotations around the $y'$ and $x' \,(i.e.\,Body)\, axes$. But we need to express everything in terms of rotations around the $x$ and $y$ axes of the LAB.

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix}$$

rotation around $x$-axis expressed relative to LAB basis.

and

$$R_y(\phi) = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To achieve this we observe that $R_x(\theta)$ and $R_x(\theta)$ are related by a change of basis:

$R_x(\theta) R_y(\phi) b = R_y(\phi) R_x(\theta) b$ components of $b$ relative to LAB frame.

components of $R(\theta) b$ relative to LAB frame.

This holds for all vectors $b$. It follows that

$$R_x(\theta) R_y(\phi) = R_y(\phi) R_x(\theta)$$
Concluding comment:
The rotation axes of the product rotation
\[ P = R_z(\Phi) R_x(\Theta) R_z(\Psi) \]
refers to the \textit{BODY} axes, not to the \textit{LAB} axes, as in our development.