

Lecture 3

Euclidean geometry vs. Lorentz geometry

[F-W 5.2, 5.3 (1.1, 1.6)] Lorentz transformations [T-W L.3, L.4,
L.5, L.6]
Euclidean vs. (1.8, 1.9)

It is difficult to overstate the importance of the principle of the invariance of the interval. It is the basis for the geometry of spacetime, just as the invariance of the Euclidean distance is the basis for the geometry of space. To highlight this importance, let us consider several examples ^(geometrical) where we compare the relation between different Euclidean frames (coordinate systems) with the geometric relations between different inertial frames.

Example 1: Pair of frames

- a) In Euclidean space two different coordinate systems can be used to determine the distance between two points

$$(\Delta x)^2 + (\Delta y)^2 = (\overline{\Delta x})^2 + (\overline{\Delta y})^2 \equiv (\Delta s)^2$$

If one uses incommensurate units along different axes, one must introduce a conversion factor, say k , in order to obtain

to obtain

$$(\Delta s)^2 = (\Delta x)^2 + (k \Delta y')^2$$

The unbarred coordinate differences are related to the barred one by means of a Euclidean rotation.

- b) In spacetime two different frames can be used to determine the interval

$$(\Delta t)^2 - (\Delta x)^2 = (\overline{\Delta t})^2 - (\overline{\Delta x})^2 \equiv (\Delta s)^2$$

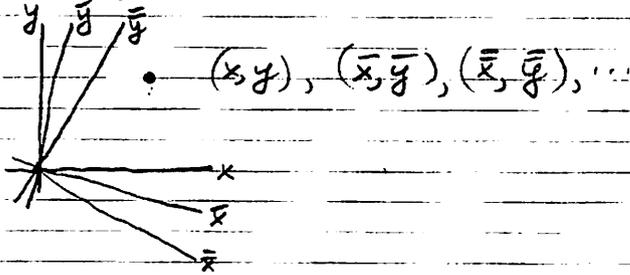
The respective coordinate separations are in general different in the two frames:

$$\Delta x \neq \overline{\Delta x} \quad \text{and} \quad \Delta t \neq \overline{\Delta t}$$

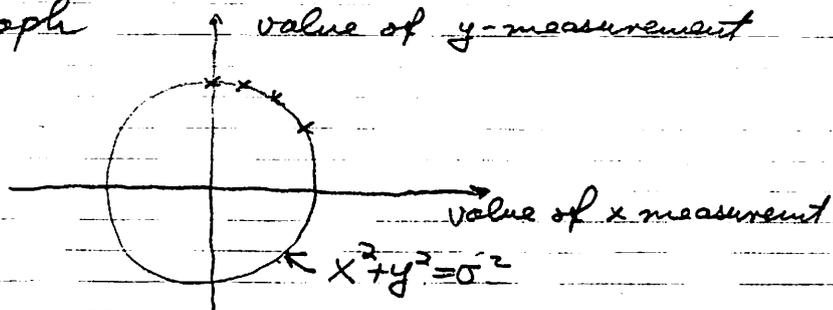
However they are related, as we shall see below, by a Lorentz rotation.

Example 2: Many frames

- a) Let the position of the same point in Euclidean space be measured by many surveyors, that are rotated relative to each other around the origin



Let us plot their measurements on a graph



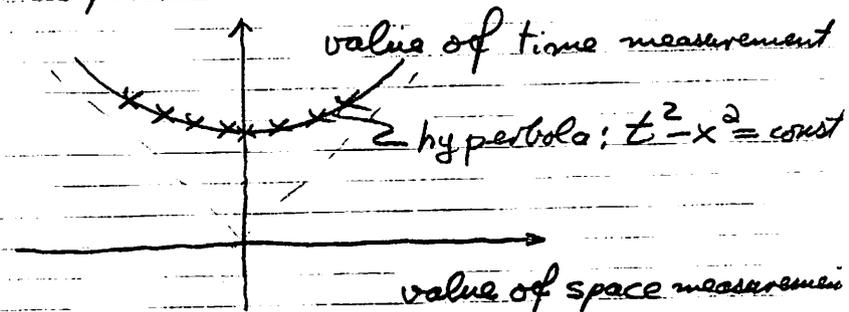
One obtains a circle given by $x^2 + y^2 = r^2$

- b) Similarly consider many Lorentz observers, all moving in the x -direction relative to each other. Let them observe two events, one at time $t = \bar{t} = \bar{\bar{t}} = \dots = 0$, $x = \bar{x} = \bar{\bar{x}} = \dots = 0$, the other at $(\Delta x, \Delta t)$, $(\bar{\Delta x}, \bar{\Delta t})$, $(\bar{\bar{\Delta x}}, \bar{\bar{\Delta t}})$, \dots

The invariance of the interval implies

$$(\Delta t)^2 - (\Delta x)^2 = (\bar{\Delta t})^2 - (\bar{\Delta x})^2 = \dots = (\Delta \tau)^2$$

Let us plot their measurements

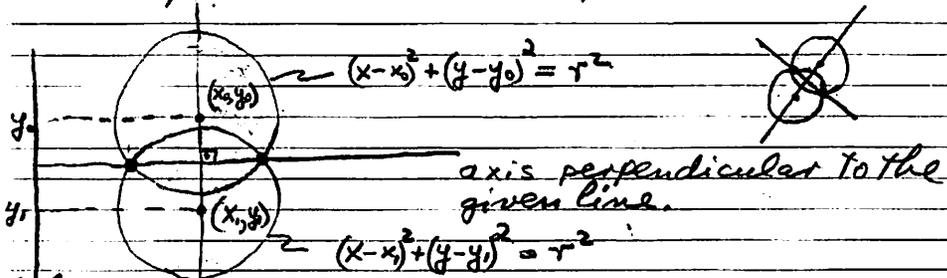


One obtains a hyperbola $t^2 - x^2 = \tau^2 = \text{const}$

3.5

Example 3: Perpendicular axis.

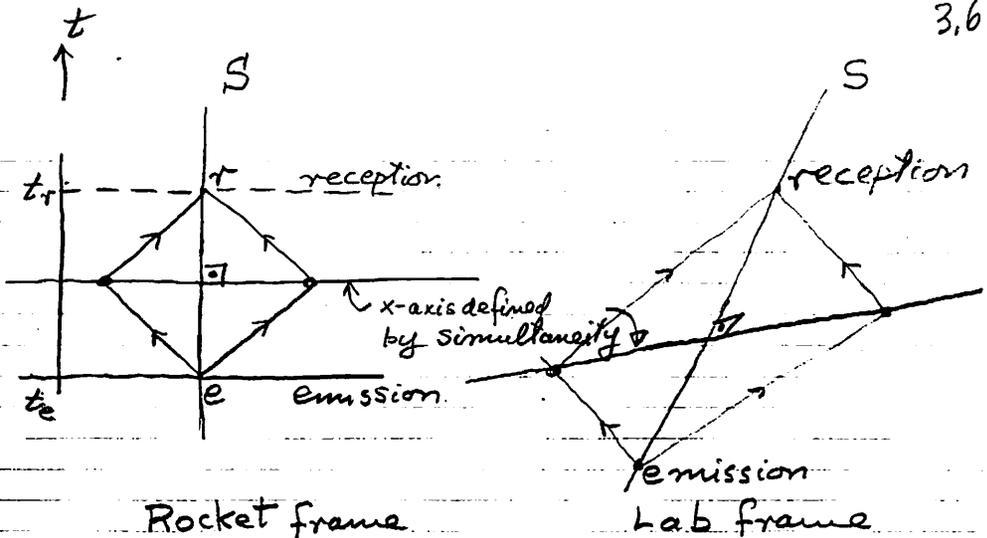
a) Given two points on a straight line in Euclidean space, the locus of points equidistant from those two points is an axis perpendicular to the line



Only two intersecting loci of equal distance from two respective centers determine an axis perpendicular to the given line

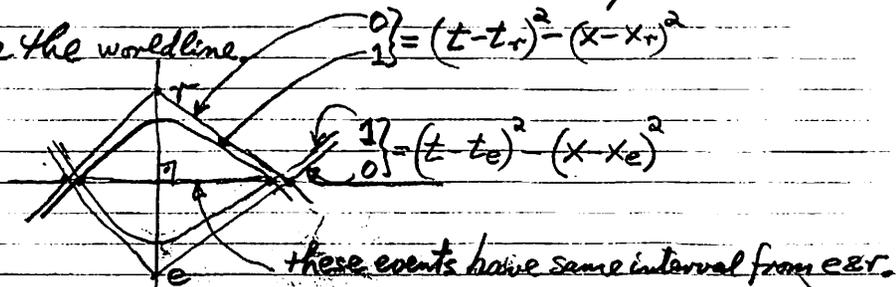
b) Given two events on a straight worldline in space time, the locus of events equidistant from these two events is an axis perpendicular to or simultaneous relative to the given worldline

3.6

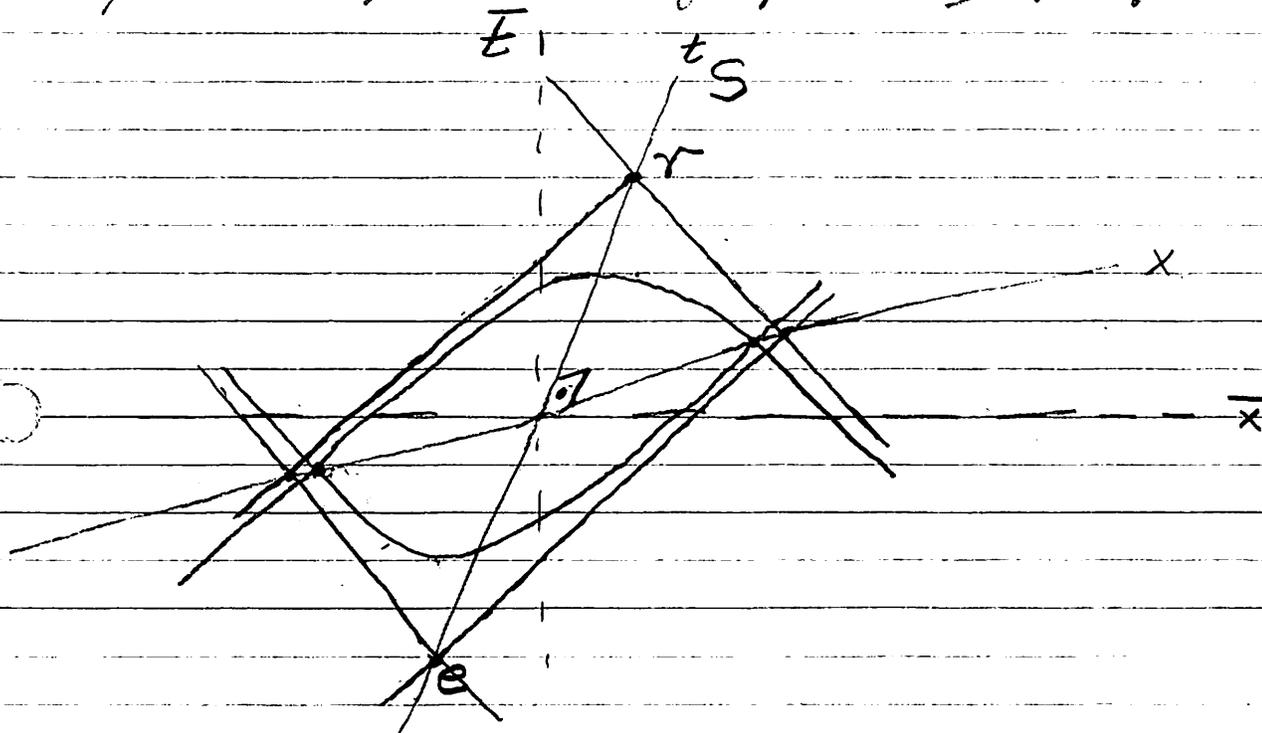


Rocket frame
Use two reflected radar pulses to establish the locus of events simultaneous relative to the straight worldline of the radar station.

The locus of simultaneous events is perpendicular to the worldline because each of these events is the same invariant interval from the emission and reception event on the worldline.



The perpendicular axis passes through the events of intersection pairs of "pseudo circles" (=hyperbolas, including their degenerate limit, the intersecting past and future asymptotes) of equal radii.



$$S : (t, x)$$

$$\bar{S} : (\bar{t}, \bar{x})$$

Example 4: Transformation between a pair of frames

I. In Euclidean space:

1. Recall the rotation transformation between a pair of Euclidean frames



$$\begin{aligned} x &= \bar{x} \cos \theta + \bar{y} \sin \theta \\ y &= -\bar{x} \sin \theta + \bar{y} \cos \theta \end{aligned} \quad \text{i.e.} \quad \begin{pmatrix} x \\ y \end{pmatrix} = T_\theta \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

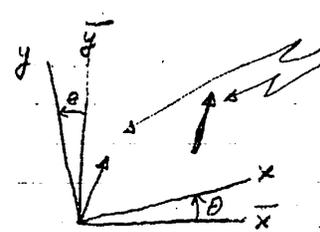
COORD. COMP. RELATIVE TO NEW BASIS } OR } COORD. COMP. RELATIVE TO OLD BASIS

$$\{ \underline{a} \} = T_\theta \{ \bar{\underline{a}} \}$$

This transformation is characterized by

- a) Linearity
- b) trigonometric coefficients.

a) Why Linearity? Because the Euclidean plane is homogeneous, i.e. its (geometrical) properties are the same at all points. In particular, a rotation transformation, such as T_θ , around one reference point has the same effect on the coordinates of a vector at one point as on those of a parallel translated vector at another point.



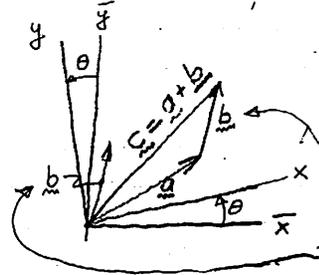
of parallel vectors whose coordinate components relative to the $x-y$ and $\bar{x}-\bar{y}$ coord. system are the same.

The linearity of T_θ follows from this homogeneity when T_θ is applied to the coordinates of three vectors $\underline{a}, \underline{b}, \underline{c}$

$$\begin{aligned} \underline{a} &: \{ \underline{a} \} = T_\theta \{ \bar{\underline{a}} \} \\ \underline{b} &: \{ \underline{b} \} = T_\theta \{ \bar{\underline{b}} \} \\ \underline{c} &: \{ \underline{c} \} = T_\theta \{ \bar{\underline{c}} \} \end{aligned}$$

which make up a closed triangle

$$\underline{a} + \underline{b} = \underline{c} \quad \{ \bar{\underline{a}} \} + \{ \bar{\underline{b}} \} = \{ \bar{\underline{c}} \} \quad (*)$$



We have a (closed) triangle because Euclid considers the two parallel vectors and

as equivalent, i.e. geometrically the same. This equivalence of parallel vectors is a property characteristic of the

Euclidean plane.

The closedness of triangle $a b c$ is preserved under a rotation T_θ of the coordinate system. This is expressed by

$$\underline{a} + \underline{b} = \underline{c} : T_\theta(\{\underline{a}\}) + T_\theta(\{\underline{b}\}) = T_\theta(\{\underline{c}\})$$

Using Eq. (*) on P.3.9 we have

$$T_\theta(\{\underline{a}\}) + T_\theta(\{\underline{b}\}) = T_\theta(\{\underline{a}\} + \{\underline{b}\}) \quad (**)$$

i.e. T_θ is linear!

To obtain this result we used the fact that in the Euclidean plane:

- (i) parallel vectors are equivalent
(i.e. they form an equivalence class, which is what is meant by a vector in linear algebra)
- (ii) parallel vectors remain parallel when their description is changed by rotating the coordinate axes:
 $\{\underline{a}\} \rightarrow \underline{a} = T(\{\underline{a}\})$
- (iii) the description of a triangle as being closed, Eq. (*) and (**), remain unchanged under rotation of coordinate axes.

As a consequence the composite is also linear:
transformation

$$\begin{array}{c} T \\ \text{LINEAR} \\ \circ \end{array} \quad \begin{array}{c} T_{\theta_1} \\ \uparrow \\ \text{linear} \end{array} \quad \begin{array}{c} T_{\theta_2} \\ \uparrow \\ \text{linear} \end{array} = \begin{array}{c} T_{\theta_1 + \theta_2} \\ \swarrow \\ \text{also linear} \end{array}$$

i.e. all transformations have the same form

- b) trigonometric coefficients $\{\cos\theta, \sin\theta\}$

Why? To guarantee invariance of the squared distance

$$x^2 + y^2 = \bar{x}^2 + \bar{y}^2$$

Summary

Euclidean rotation:

- (i) Why linear? Our observations about reality (Euclidean plane) demand it. In particular, the form of the rotation law must be the same around any origin.
- (ii) Why \sin & \cos ? Invariance of the squared distance demands it.

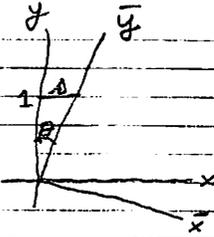
2) Instead of θ , use

$$\boxed{s = \frac{\sinh \theta}{\cosh \theta} = \tanh \theta}$$

the slope, to characterize the transformation law

$$x = \frac{\bar{x} + s\bar{y}}{\sqrt{1-s^2}}$$

$$y = \frac{-\bar{x}s + \bar{y}}{\sqrt{1-s^2}}$$

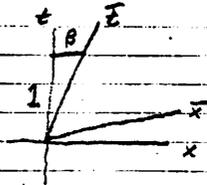


(II. In space-time we introduce a transformation law with the same requirements: (i) linearity (because of P. & R.) plus (ii) the invariance of the interval give rise to

$$\begin{aligned} t &= \bar{t} \cosh \theta + \bar{x} \sinh \theta \\ x &= \bar{t} \sinh \theta + \bar{x} \cosh \theta \end{aligned} \quad \text{or} \quad \begin{pmatrix} t \\ x \end{pmatrix} = \Lambda \begin{pmatrix} \bar{t} \\ \bar{x} \end{pmatrix}$$

If instead of the parameter θ one uses the slope

$$\beta = \frac{\sinh \theta}{\cosh \theta} = \tanh \theta$$



then $t = \frac{\bar{t} + \beta \bar{x}}{\sqrt{1-\beta^2}}$

$$x = \frac{\beta \bar{t} + \bar{x}}{\sqrt{1-\beta^2}}$$

Here the slope is simply the relative velocity: because $\bar{x} = 0$ yields the velocity of the spatial origin of \bar{S} :

$$\text{velocity} = \frac{x}{t} = \frac{\sinh \theta}{\cosh \theta} = \beta.$$

This transformation of the set of coordinates relative to one frame to that relative to another frame is called a Lorentz transformation.